Lecture VII

Linear Models

We have now learned how to solve all the first-order equations that this course has to offer. Before moving on to higher-order equations and systems of equations, we will learn some applications of first-order ODEs in science.

Growth & Decay Models

$$\frac{\mathrm{d}x}{\mathrm{d}t} = kx \quad \text{subject to} \quad x(t_0) = x_0 \tag{1}$$

This equation represents a model for exponential growth (or decay), typically of a population of bacteria or small animals, of concentrations of some product in a first-order reaction, or even of the half-life of some radioactive element. Here x would represent the variable that is growing or decaying, say the number of bacteria in a Petri dish, t is time, and k is a constant of proportionality signifying growth if it is positive or decay if it is negative. The model is suggesting that the rate of change of x per unit time ($\mathrm{d}x/\mathrm{d}t$) is directly proportional to x at the "present" time t (i.e. $x = x(t)$).

EXAMPLE

An invasive species is a living organism that is introduced into an ecosystem (other than its native ecosystem) that has a direct (often negative) impact on the ecosystem itself. These invasive species tend to be introduced by humans due to transport of goods (insects and small animals might find their way on ships for example), release of some pets to the wild, or they may be purposefully introduced in the hopes that the introduced species will prey on or out-compete an already present invasive species. When an invasive species is introduced to a new ecosystem, it will tend to have no natural predators in that system, allowing the species to initially grow exponentially. A species of rat was introduced to an island in which it has no natural predators (as happened on Gough island). Initially, 4 rats were introduced to the island after hitching a ride on a ship. If after 1 year, the population of rats on the island grew to 40, how many rats would there be an the island after 2 years assuming a model for exponential growth.

Let n represent the number of rats. We have the initial condition $n(0) = 4$ and an additional condition $n(1) = 40$. The model is a separable equation, and can easily be solved,

$$\frac{\mathrm{d}n}{\mathrm{d}t} = kn \rightarrow n(t) = ce^{kt}.$$

Using the two conditions we find

$$\begin{aligned} n(0) = 4 \quad &\rightarrow \quad c = 4 \\ n(1) = 40 \quad &\rightarrow \quad k = \ln(10). \end{aligned}$$

The equation then reduces to

$$n(t) = 4e^{\ln(10)t} = 4\left(10^t\right),$$

after which we can determine the amount of rats that will be on the island in 2 years by setting $t = 2$, namely 400 rats!

Newton's Law of Cooling or Warming

$$\frac{\mathrm{d}T}{\mathrm{d}t} = h(T - T_\infty) \tag{2}$$

This equation represents a model for cooling or heating of a substance by convection, with T being the temperature, t the time, T_∞ the temperature of the surrounding medium (such as air or some other fluid), and h the constant of proportionality. Here we are suggesting that the rate of change of the temperature of a substance surrounded by some fluid is proportional to the temperature of the substance and to that of the surrounding medium (which is assumed to be constant).

EXAMPLE

A bar of hot metal, initially at 300^{o}C, is quenched in a very large bath of cold swirling water at 4^oC. In 30 seconds, the temperature of the metal had already dropped to 60^{o}C. How long will it take the metal bar to reach 20^{o}C?

The temperature of the medium, the cold bath of water, is $T_\infty = 4$ and we have the two conditions $T(0) = 300$ and $T(30) = 60$. The governing differential equation, Newton's law of cooling, is separable and can easily be solved as

$$\frac{\mathrm{d}T}{\mathrm{d}t} = h(T - T_\infty) \rightarrow T(t) = ce^{ht} + T_\infty.$$

Using the two conditions we find

$$\begin{aligned} T(0) = 300 \quad &\rightarrow \quad c = 296 \\ T(30) = 60 \quad &\rightarrow \quad h = \tfrac{1}{30}\ln\left(\tfrac{7}{37}\right). \end{aligned}$$

The equation then reduces to

$$T(t) = 296e^{\frac{t}{30}\ln\left(\frac{7}{37}\right)} + 4 = 296\left(\frac{7}{37}\right)^{\frac{t}{30}} + 4,$$

after which we can determine the time it will take to cool the metal bar down to 20^{o}C, namely 52.6 seconds.

Mixing

$$\frac{\mathrm{d}m}{\mathrm{d}t} = R_{in} - R_{out} \tag{3}$$

This is really an equation of conservation of mass. The equation is saying that the rate of change of the mass m of some substance in some control volume is dependent upon the rates of influx (R_{in}) and outflux (R_{out}) of that substance relative to the control volume.

EXAMPLE

A swimming pool is filled with 60,000 L of water and initially contains no chlorine. A solution of 1.5 mg/L of chlorine is pumped into the pool at a rate of 300 L/min. The well-mixed chlorine-water solution of the pool is pumped out at the same rate. If the concentration of chlorine in the pool is to be 1 mg/L, how long will it take before the pool has attained the correct concentration? What would R_{out} look like if the solution were being pumped out at

250 L/min? What about 380 L/min?

The mass flow rate of chlorine entering the pool is $R_{in} = (1.5 \text{ mg/L})(300 \text{ L/min}) = 450 \text{ mg/min}$ and the mass flow rate of chlorine exiting the pool $R_{out} = (\frac{m}{60{,}000} \text{ mg/L})(300 \text{ L/min}) = \frac{m}{200} \text{ mg/min}$. We would like to know how long it will take for the mass of chlorine in the pool to reach (1 mg/L)(60,000 L) = 60,000 mg. The differential equation and solution is given by

$$\frac{\mathrm{d}m}{\mathrm{d}t} = 450 - \frac{m}{200} \rightarrow m(t) = 90,000 + ce^{-\frac{t}{200}}.$$

Since there is no chlorine in the pool at the beginning, $m(0) = 0$ giving $c = -90,000$, and the solution becomes

$$m(t) = 90,000\left(1 - e^{-\frac{t}{200}}\right).$$

Therefore, we can determine that it will take 220 minutes to fill the pool to the required concentration.

In the case that we are pumping the solution out at 250 L/min, we have to consider that the volume in the pool will increase with time since we are pumping the solution out at a slower rate than we are pumping it in (250 L/min vs. 300 L/min). So after a time t, the volume in the tank will be $V = 60,000 + (300 - 250)t = 60,000 + 50t$, and therefore $R_{out} = (\frac{m}{60{,}000+50t} \text{ mg/L})(250 \text{ L/min}) = \frac{5m}{1200+t} \text{ mg/min}$. Similarly, in the case that we are pumping the solution out at 380 L/min, the volume in the pool will decrease as $V = 60,000 + (300 - 380)t = 60,000 - 80t$ and so $R_{out} = (\frac{m}{60{,}000-80t} \text{ mg/L})(380 \text{ L/min}) = \frac{19m}{3000+4t} \text{ mg/min}$.

RL & RC Circuits

$$L\frac{\mathrm{d}i}{\mathrm{d}t} + Ri = E(t) \quad \text{or} \quad R\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{1}{C}q = E(t) \tag{4}$$

The model on the left is the application of Kirchhoff's Second Law on an RL circuit (a circuit with a voltage source $E(t)$, a resistor R, and an inductor L), and describes the behaviour of the current (i) with time (t) as we go around the circuit. On the right, we have the application of Kirchhoff's Second Law on an RC circuit (a circuit with a voltage source $E(t)$, a resistor R, and a capacitor C), and describes the behaviour of charge (q) with time (t) as we go around the circuit.

Lecture Problems (§2.7): 6, 15, 23

Tutorial Problems (§2.7): 5, 11, 14, 21

Suggested Problems (§2.7): 3, 13, 17

BONUS NOTES

None.

REFERENCES

Zill, D. G., & Wright, W. S. (2014). *Advanced Engineering Mathematics* (5th ed.). Burlington, MA: Jones & Bartlett Learning.