Lecture XVIII

Power Series Solutions

We will now look at solving the general homogeneous linear second-order ODE

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0, \tag{1}$$

or in particular, in standard form

$$y'' + P(x)y' + Q(x)y = 0. \tag{2}$$

DEFINITION: Ordinary and Singular Points

A point x_0 is said to be an **ordinary point** of the differential equation (1) if both $P(x)$ and $Q(x)$ in the standard form (2) are analytic at x_0. A point that is not an ordinary point is said to be a **singular point** of the equation.

THEOREM: Existence of a Power Series Solution

If $x = x_0$ is an ordinary point of the differential equation (1), we can always find two linearly independent solutions in the form of a power series centered at x_0; that is, $y = \sum_{n=0}^{\infty} c_n(x-x_0)^n$. A series solution converges at least on some interval defined by $|x-x_0| < R$, where R is the distance from x_0 to the closest singular point.

Finding a power series solution to the homogeneous linear second-order ODE amounts to assuming the solution to be $y = \sum_{n=0}^{\infty} c_n(x-x_0)^n$, substituting it into the ODE and extracting the two linearly independent solutions $y_1(x)$ and $y_2(x)$.

EXAMPLE

$$\text{Solve} \quad y'' + x^3 y = 0.$$

We see that $x_0 = 0$ is an ordinary point of the ODE. We will therefore substitute $y =$

$\sum_{n=0}^{\infty} c_n x^n$ giving

$$\left(\sum_{n=0}^{\infty} c_n x^n\right)'' + x^3 \left(\sum_{n=0}^{\infty} c_n x^n\right) = 0$$
$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + x^3 \left(\sum_{n=0}^{\infty} c_n x^n\right) = 0$$
$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+3} = 0$$
$$2c_2 + 6c_3 x + 12c_4 x^2 + \underbrace{\sum_{n=5}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} + \underbrace{\sum_{n=0}^{\infty} c_n x^{n+3}}_{k=n+3} = 0$$
$$2c_2 + 6c_3 x + 12c_4 x^2 + \sum_{k=3}^{\infty}(k+1)(k+2)c_{k+2} x^k + \sum_{k=3}^{\infty} c_{k-3} x^k = 0$$
$$2c_2 + 6c_3 x + 12c_4 x^2 + \sum_{k=3}^{\infty} \left[c_{k-3} + (k+1)(k+2)c_{k+2}\right] x^k = 0.$$

Now, we ought to note that the left-hand side of the equation (a polynomial of infinite extent) should equal the right-hand side of the equation (zero), therefore we have

$$2c_2 = 0 \to c_2 = 0$$
$$6c_3 = 0 \to c_3 = 0$$
$$12c_4 = 0 \to c_4 = 0$$
$$c_{k-3} + (k+1)(k+2)c_{k+2} = 0 \quad \forall k \to c_{k+2} = \frac{c_{k-3}}{(k+1)(k+2)}.$$

If we looked at the values for the constants one by one, we see that we can express the solution using only two constants c_0 and c_1,

$$
\begin{array}{lll}
k=3, & c_5 = \frac{c_0}{4\cdot 5} & \\
k=4, & c_6 = \frac{c_1}{5\cdot 6} & \\
k=5, & c_7 = 0 & \leftarrow c_2 = 0 \\
k=6, & c_8 = 0 & \leftarrow c_3 = 0 \\
k=7, & c_9 = 0 & \leftarrow c_4 = 0 \\
k=8, & c_{10} = \frac{c_0}{4\cdot 5\cdot 9\cdot 10} & \\
k=9, & c_{11} = \frac{c_1}{5\cdot 6\cdot 10\cdot 11} & \\
k=10, & c_{12} = 0 & \leftarrow c_7 = 0 \\
k=11, & c_{13} = 0 & \leftarrow c_8 = 0.
\end{array}
$$

Substituting back into the original assumption, $y = \sum_{n=0}^{\infty} c_n x^n$, the general solution is clearly the sum of two linearly independent series solutions,

$$y = c_0\left(1 + \frac{x^5}{4\cdot 5} + \frac{x^{10}}{4\cdot 5\cdot 9\cdot 10} + \cdots\right) + c_1\left(x + \frac{x^6}{5\cdot 6} + \frac{x^{11}}{5\cdot 6\cdot 10\cdot 11} + \cdots\right).$$

Lecture Problems (§5.1): 31

Tutorial Problems (§5.1): 26, 30

Suggested Problems (§5.1): 17, 19, 21, 25

BONUS NOTES

None.

REFERENCES

Zill, D. G., & Wright, W. S. (2014). *Advanced Engineering Mathematics* (5th ed.). Burlington, MA: Jones & Bartlett Learning.