Lecture XV

Nonlinear Equations

Up to now, we have learned almost all the "clean" methods of solving ODEs that this course has to offer before using power series solutions. In this lecture, we look at the last of these "clean" methods. We will look at the general second-order nonlinear ODE $F(x, y, y', y'') = 0$ in two cases, one when the dependent variable y is missing and the other when the independent variable x is missing. In either case, the ODE can sometimes be reduced to a first-order ODE using the substitution $u = y'$.

EXAMPLE

$$\text{Solve} \quad y'' = \frac{4}{\sec^2(y')}.$$

This is a second-order nonlinear ODE where **the dependent variable y is missing**. We will use the substitutions

$$\frac{\mathrm{d}y}{\mathrm{d}x} = u \quad \text{and} \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}u}{\mathrm{d}x},$$

giving

$$u' = \frac{4}{\sec^2(u)}.$$

This equation is now separable and we can solve for u

$$u = \arctan(4x + c_1),$$

and now since $y' = u$, integration of the solution for u will give

$$y = \left(x + \frac{c_1}{4}\right)\arctan(4x + c_1) - \frac{1}{8}\ln\left|16x^2 + 1 + c_1(c_1 + 8x)\right| + c_2.$$

* Note that, for both cases discussed in this lecture, although it is always possible to solve the differential equation for u (i.e. for $\mathrm{d}y/\mathrm{d}x$) by separation of variables (provided the integration is possible), upon integration we might still obtain an ODE that cannot be solved using the methods we have learned (it might not even be possible to isolate $\mathrm{d}x$ or $\mathrm{d}y$).

EXAMPLE

$$\text{Solve} \quad \frac{y''}{y} = -2(y')^3.$$

This is a second-order nonlinear ODE where **the independent variable x is missing**. We will use the substitutions

$$\frac{\mathrm{d}y}{\mathrm{d}x} = u \quad \text{and} \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}x} = u\frac{\mathrm{d}u}{\mathrm{d}y},$$

giving

$$\frac{1}{y}u\frac{\mathrm{d}u}{\mathrm{d}y} = -2u^3.$$

Again, this equation is separable and we can solve for u

$$u = \frac{1}{y^2 + c_1}.$$

We can now solve for y from

$$\frac{\mathrm{d}y}{\mathrm{d}x} = u \rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{y^2 + c_1},$$

which is also separable, yielding

$$x = \frac{1}{3}y^3 + c_1 y + c_2.$$

Lecture Problems (§3.7): 4, 5, 8

Tutorial Problems (§3.7): 2, 6

Suggested Problems (§3.7): 1, 3, 7

BONUS NOTES

None.

REFERENCES

Zill, D. G., & Wright, W. S. (2014). *Advanced Engineering Mathematics* (5th ed.). Burlington, MA: Jones & Bartlett Learning.