Lecture I

Definitions & Terminology

We begin this course with perhaps the most important question: What is a Differential Equation?

DEFINITION: Differential Equation

"An equation containing the derivatives of one or more dependent variables, with respect to one or more independent variables, is said to be a **Differential Equation** (**DE**). There are two **types** of differential equations. If a differential equation contains only ordinary derivatives of one or more functions with respect to a **single** independent variable it is said to be an **Ordinary Differential Equation** (**ODE**). An equation involving only partial derivatives of one or more functions of two or more independent variables is called a **Partial Differential Equation** (**PDE**)." (Zill & Wright, 2014)

Take for example the differential equation,

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \sin(y) = e^x.$$

This is an ordinary differential equation since it contains only ordinary derivatives ($\mathrm{d}/\mathrm{d}x$) of the dependent variable $y(x)$ with respect to the single independent variable x. In contrast,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

is a partial differential equation since it involves only partial derivatives ($\partial/\partial x$ & $\partial/\partial y$) of the function $u(x, y)$ with respect to the independent variables x and y.

Why should we care? What does this course have to do with our lives? Well, as it turns out, other than the very fundamental physics you learned back in kindergarten, the laws of nature are best described by differential equations. Whether it be the true form of Newton's second law governing the motion of everyday objects, the rate equations governing chemical reactions, Maxwell's equations governing electromagnetism, or the Navier-Stokes equations governing the flow of fluids, differential equations are everywhere and they are an absolute necessity in science and engineering. In this course, you are taking your first steps in learning how to solve central problems in engineering. We will be focusing on ordinary differential equations. You will approach partial differential equations in ENGR 311.

Before discussing more definitions and terminology to be used in this course, let us recall the common notations used for ordinary derivatives:

1) **Leibniz Notation**

$$y = y(x) \to \frac{\mathrm{d}y}{\mathrm{d}x},\ \frac{\mathrm{d}^2y}{\mathrm{d}x^2},\ \frac{\mathrm{d}^3y}{\mathrm{d}x^3},\ \frac{\mathrm{d}^4y}{\mathrm{d}x^4},\ \frac{\mathrm{d}^5y}{\mathrm{d}x^5},\ \ldots,\ \frac{\mathrm{d}^ny}{\mathrm{d}x^n}$$

2) **Prime Notation**

$$y = y(x) \to y',\ y'',\ y''',\ y^{(4)},\ y^{(5)},\ \ldots,\ y^{(n)}$$

3) **Newton Dot Notation**

$$y = y(x) \to \dot{y},\ \ddot{y},\ \dddot{y},\ \overset{(4)}{\dot{y}},\ \overset{(5)}{\dot{y}},\ \ldots,\ \overset{(n)}{\dot{y}}$$

Other than its **type** (i.e. ordinary or partial), we can classify a differential equation by its **order** and **linearity**. The **order** of a differential equation is the order of the highest derivative in the equation. For example,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 5\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^3 - 4y = e^x$$

has order 2 since the highest order derivative in the equation is $\mathrm{d}^2y/\mathrm{d}x^2$. **Do not mistake the order of the differential equation with the highest power (or exponent) appearing in the equation!**

We may express ordinary differential equations in their **general form**, making use of the prime notation for derivatives,

$$F\left(x, y, y', y'', y''', y^{(4)}, ..., y^{(n)}\right) = 0 \quad \text{where} \quad y = y(x) \tag{1}$$

or in their **normal form** (not always possible)

$$y^{(n)} = f\left(x, y, y', y'', y''', y^{(4)}, ..., y^{(n-1)}\right). \tag{2}$$

An nth-order ODE is said to be **linear** if F is linear in $y, y', y'', ..., y^{(n)}$ (Zill & Wright, 2014), namely

$$a_n(x)\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + a_{n-1}(x)\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \cdots + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y - g(x) = 0. \tag{3}$$

where the $a_j(x)$, $j = 1, 2, ..., n$ are arbitrary functions of x and at least one $a_j(x) \neq 0$ for $j \geq 1$ (otherwise we don't even have a differential equation). It is often useful to express a linear ODE in **standard form** after dividing through by $a_n(x)$:

$$\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + b_{n-1}(x)\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \cdots + b_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + b_0(x)y = h(x). \tag{4}$$

The course is based upon solving ODEs, so we are faced with the question: What does the solution of an ODE even mean?

DEFINITION: Solution of an ODE

"Any function ϕ, defined on an interval I and possessing at least n derivatives that are continuous on I, which when substituted into an nth-order ordinary differential equation reduces the equation to an identity, is said to be a **solution** of the equation on the interval." (Zill & Wright, 2014)

Basically, for an ordinary differential equation in the dependent variable y, a solution of the differential equation $y = \phi(x)$ will "work" when we plug it into the original equation. Let's show this with an example.

EXAMPLE

Verify that $y'' - 5y' + 6y = 0$ has the solution $y = c_1 e^{2x} + c_2 e^{3x}$, where c_1 and c_2 are arbitrary constants. Let's plug the solution into the ODE:

$$(c_1 e^{2x} + c_2 e^{3x})'' - 5(c_1 e^{2x} + c_2 e^{3x})' + 6(c_1 e^{2x} + c_2 e^{3x}) = 0$$
$$(4c_1 e^{2x} + 9c_2 e^{3x}) - 5(2c_1 e^{2x} + 3e^{3x}) + 6(c_1 e^{2x} + c_2 e^{3x}) = 0$$
$$c_1(4 - 10 + 6)e^{2x} + c_2(9 - 15 + 6)e^{3x}$$
$$0 = 0.$$

This is an example of an **explicit** solution (i.e. a solution that can be represented as $y = \phi(x)$), however a solution could also be **implicit** (i.e. a solution that cannot be expressed in the form $y = \phi(x)$, but rather only as $\phi(x, y) = 0$).

Thinking deeply about ODEs, we see that it is not at all obvious to answer the question: If we have a solution, is it always valid? Unfortunately, no, a solution may not necessarily be valid for all values of the independent variable. We have to always keep in mind the **interval of definition**, **interval of validity** or **domain** of a solution to a differential equation. For the example above, the domain of the solution is $x \in (-\infty, +\infty)$.

Lecture Problems (§1.1): 3, 12, 16
Tutorial Problems (§1.1): 4, 5, 13, 24
Suggested Problems (§1.1): 1, 2, 6, 10, 11, 23

BONUS NOTES

1) A good way to verify that you have described the correct domain of your solution is to actually graph your solution. You can use an online graphing calculator such as desmos: https://www.desmos.com/calculator

2) Knowing how to do both ordinary and partial derivatives and especially how to do integrals is an absolute necessity for this course. You can always double check your derivatives and integrals using online tools available by Wolfram|Alpha:
http://www.wolframalpha.com/examples/Calculus.html

REFERENCES

Zill, D. G., & Wright, W. S. (2014). *Advanced Engineering Mathematics* (5th ed.). Burlington, MA: Jones & Bartlett Learning.