Lecture II

Initial Value Problems

Typically, in scientific problems, a differential equation will be accompanied by an initial condition. That is to say, at some specific instant or location, the solution is known to have a specific value and we can use this to determine the constant of integration resulting from the solution of the differential equation. We will also learn that, in the case of higher-order ODEs, a differential equation can be accompanied by boundary conditions. In the case of a first-order ODE, an initial value problem would be formulated as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y) \quad \text{subject to} \quad y(x_0) = y_0, \tag{1}$$

however in the case of an nth-order ODE, we would require n initial conditions, one for each derivative of y except $y^{(n)}$, i.e.

$$\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = f\left(x, y, y', y'', ..., y^{(n-1)}\right) \quad \text{with} \quad y(x_0) = y_0, y'(x_0) = y_0', ..., y^{(n-1)}(x_0) = y_0^{(n-1)}. \tag{2}$$

EXAMPLE

The solution to the ODE $y'' - 3y' + 2y = 0$ is $y = c_1 e^x + c_2 e^{2x}$. Determine the value of the constant subject to the initial conditions $y(0) = 1$ and $y'(0) = 0$. Let's plug in therefore the two initial conditions to determine the constants:

$$\begin{aligned} y(0) = 1 \quad &\rightarrow \quad c_1 + c_2 = 1 \\ y'(0) = 0 \quad &\rightarrow \quad c_1 + 2c_2 = 0 \\ c_1 = 2 \quad &\text{and} \quad c_2 = -1. \end{aligned}$$

The solution to the initial value problem

$$y'' - 3y' + 2y = 0 \quad \text{subject to} \quad y(0) = 1 \quad \text{and} \quad y'(0) = 0$$

is therefore $y = 2e^x - e^{2x}$.

How do we know that the solution to an initial value problem is unique though? Can there be other solutions? In fact, it may happen that there are two (or perhaps more) solutions to an initial value problem. Take the example $3y' = y^{-2}$ subject to $y(0) = 0$. Both the functions $y = 0$ and $y = \sqrt[3]{x}$ satisfy the initial value problem. We do have however, the following existence and uniqueness theorem for first-order initial value problems of the form (1).

THEOREM: Existence of a Unique Solution

Let R be a rectangular region in the xy-plane defined by $a \leq x \leq b$, $c \leq y \leq d$, that contains the point (x_0, y_0) in its interior. If $f(x, y)$ and $\partial f/\partial y$ are continuous on R, then there exists some interval $I_0 : (x_0 - h, x_0 + h), h > 0$, contained in $[a, b]$, and a unique function $y(x)$ defined on I_0 that is a solution of the initial value problem (1)." (Zill & Wright, 2014)

Lecture Problems (§1.2): 8, 11

Tutorial Problems (§1.2): 9, 13, 14

Suggested Problems (§1.2): 7, 12, 18

BONUS NOTES

None.

REFERENCES

Zill, D. G., & Wright, W. S. (2014). *Advanced Engineering Mathematics* (5th ed.). Burlington, MA: Jones & Bartlett Learning.