Lecture XI

Homogeneous Linear Equations with Constant Coefficients

Again, let us consider the homogeneous linear second-order ODE, however this time with constant coefficients

$$ay'' + by' + cy = 0. \tag{1}$$

For such a case, we will try to plug in a solution of the form $y = e^{mx}$ and see what results, i.e.

$$a\left(e^{mx}\right)'' + b\left(e^{mx}\right) + c\left(e^{mx}\right) = 0$$

$$am^2e^{mx} + bme^{mx} + ce^{mx} = 0$$

$$am^2 + bm + c = 0.$$

It turns out we are left with a quadratic equation in m, called the **auxiliary equation**. If we solve this quadratic equation, we will have one or two values for m and so one or two corresponding solutions (however, we know there should be two solutions in the fundamental set). The cases are, classically,

1) $b^2 - 4ac > 0 \to m_1$ and m_2 are distinct.
2) $b^2 - 4ac = 0 \to m_1$ and m_2 are real and equal.
3) $b^2 - 4ac < 0 \to m_1$ and m_2 are complex conjugates.

Let us consider each case one by one. In the first case, m_1 and m_2 are distinct, therefore both the solutions $y_1 = c_1e^{m_1x}$ and $y_2 = c_2e^{m_2x}$ must form the general solution.

$$\text{Case 1: Distinct Real Roots} \to y = c_1e^{m_1x} + c_2e^{m_2x}. \tag{2}$$

In the case of the equation $y'' - k^2y = 0$, which has two distinct real roots, it is also common to write the solution in the form

$$y = c_1\sinh(kx) + c_2\cosh(kx).$$

In the second case, both the roots are equal, i.e. $m_1 = m_2 = m$. In this case, we know that $y_1 = c_1e^{mx}$ must be a solution, however we are missing the second solution. It can be obtained via the reduction of order technique, namely

$$y_2 = e^{mx} \int \frac{e^{2mx}}{e^{2mx}} \mathrm{d}x = xe^x;$$

note that in the case of a double root, the differential equation could be written as $y'' - 2my' + m^2 y = 0$, where in equation (1) we have $b = -2ma$ and $c = m^2 a$. The general solution is therefore

$$\text{Case 2: Repeated Real Roots} \rightarrow y = c_1 e^{mx} + c_2 x e^{mx}. \tag{3}$$

Finally, we arrive at the third case, however it requires more work. In this case, the roots are complex conjugates $m_{1,2} = \alpha \pm i\beta$ and so we have the general solution $y = \tilde{c}_1 e^{(\alpha+i\beta)x} + \tilde{c}_2 e^{(\alpha-i\beta)x}$, where $\tilde{c}_1$ and $\tilde{c}_2$ are complex. In practice however, we do not desire the imaginary part of the solution, so we will work to extract only the real part

$$\begin{aligned} y &= \tilde{c}_1 e^{(\alpha+i\beta)x} + \tilde{c}_2 e^{(\alpha-i\beta)x} \\ y &= e^{\alpha x} \left(\tilde{c}_1 e^{i\beta x} + \tilde{c}_2 e^{-i\beta x} \right) \\ y &= e^{\alpha x} \left[\tilde{c}_1 \left(\cos(\beta x) + i \sin(\beta x) \right) + \tilde{c}_2 \left(\cos(\beta x) - i \sin(\beta x) \right) \right] \\ y &= e^{\alpha x} \left[(\tilde{c}_1 + \tilde{c}_2) \cos(\beta x) + i (\tilde{c}_1 + \tilde{c}_2) \sin(\beta x) \right] \\ \mathrm{Re}(y) &= e^{\alpha x} \left[\mathrm{Re}(\tilde{c}_1 + \tilde{c}_2) \cos(\beta x) + \mathrm{Re}(i(\tilde{c}_1 + \tilde{c}_2)) \sin(\beta x) \right]. \end{aligned}$$

We can simply label the real parts of the constants above as c_1 and c_2 respectively, yielding the general real solution

$$\text{Case 3: Complex Conjugate Roots} \rightarrow y = e^{\alpha x} \left[c_1 \cos(\beta x) + c_2 \sin(\beta x) \right]. \tag{4}$$

These ideas in fact extend to nth-order linear ODEs with constant coefficients, namely the solutions of

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

can be described as follows. Any distinct root m_j of the auxiliary equation will have a corresponding solution of the form $y_j = c_j e^{m_j x}$, for instance, if all the n roots were distinct, the general solution would be

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}.$$

If any root m in the auxiliary equation is repeated k times (we say the root has multiplicity k), then we must include its k linearly independent solutions

$$e^{mx}, xe^{mx}, x^2 e^{mx}, ..., x^{k-1} e^{mx}.$$

In a similar fashion, all the distinct complex conjugate root pairs $\alpha_j \pm i\beta_j$ will have their corresponding solution of the form

$$y_j = e^{\alpha_j x} \left[c_{j1} \cos(\beta_j x) + c_{j2} \sin(\beta_j x) \right],$$

and for any complex conjugate roots of multiplicity k, we must include its $2k$ linearly independent solutions

$$e^{\alpha x} \cos(\beta x), xe^{\alpha x} \cos(\beta x), x^2 e^{\alpha x} \cos(\beta x), ..., x^{(k-1)} e^{\alpha x} \cos(\beta x)$$
$$e^{\alpha x} \sin(\beta x), xe^{\alpha x} \sin(\beta x), x^2 e^{\alpha x} \sin(\beta x), ..., x^{(k-1)} e^{\alpha x} \sin(\beta x).$$

EXAMPLE

$$\text{Solve} \quad y''' - y'' + 4y' - 4y = 0.$$

We must first get the auxiliary equation $m^3 - m^2 + 4m - 4 = 0$, which can be factored as $(m-1)(m^2+4) = 0$. The roots are therefore $m_1 = 1$ and $m_{2,3} = \pm 2i$ (i.e. $\alpha = 0$, $\beta = 2$). The general solution can now easily be assembled

$$y = c_1 e^x + c_2 \cos(2x) + c_3 \sin(2x).$$

Lecture Problems (§3.3): 2, 3, 16

Tutorial Problems (§3.3): 3, 7, 20, 30, 40

Suggested Problems (§3.3): 1, 5, 9, 29, 31, 33, 39, 41

BONUS NOTES

None.

REFERENCES

Zill, D. G., & Wright, W. S. (2014). *Advanced Engineering Mathematics* (5th ed.). Burlington, MA: Jones & Bartlett Learning.