Lecture XXI

Solution by Diagonalization

The homogeneous linear system $\mathbf{X}' = \mathbf{AX}$ in which each x_j' is expressed as a linear combination of $x_1, x_2, ..., x_n$ is said to be **coupled**. If the coefficient matrix $\mathbf{A}$ is diagonalizable, then the system can be uncoupled in that each x_j' can be expressed solely in terms of x_j. If the matrix $\mathbf{A}$ has n linearly independent eigenvectors then we can find a matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$, where $\mathbf{D}$ is a diagonal matrix whose entries are the eigenvalues of $\mathbf{A}$ and $\mathbf{P}$ is a matrix whose columns are the corresponding eigenvectors of $\mathbf{A}$. The procedure follows.

Given the homogeneous linear system of n ODEs

$$\mathbf{X}' = \mathbf{AX},$$

where the eigenvalues of $\mathbf{A}$ are distinctly $\lambda_1, \lambda_2, ..., \lambda_n$ with corresponding eigenvectors $\mathbf{K}_1, \mathbf{K}_2, ..., \mathbf{K}_n$, we can formulate the matrices $\mathbf{P}$ and $\mathbf{D}$ as

$$\mathbf{P} = \begin{bmatrix} | & | & & | \\ \mathbf{K}_1 & \mathbf{K}_2 & \cdots & \mathbf{K}_n \\ | & | & & | \end{bmatrix} \quad \text{and} \quad \mathbf{D} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}.$$

We can then simplify the system with the substitution $\mathbf{X} = \mathbf{PY}$

$$\mathbf{X}' = \mathbf{AX}$$

$$\mathbf{PY}' = \mathbf{APY}$$

$$\mathbf{Y}' = \mathbf{P}^{-1}\mathbf{APY}$$

$$\mathbf{Y}' = \mathbf{DY}.$$

The solution of the system $\mathbf{Y}' = \mathbf{D}\mathbf{Y}$ is obviously

$$\mathbf{Y} = \begin{bmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{bmatrix},$$

and so we can obtain the solution vector $\mathbf{X}$ then using $\mathbf{X} = \mathbf{P}\mathbf{Y}$.

Lecture Problems (§10.3): 2, 4

Tutorial Problems (§10.3): 6, 7, 8

Suggested Problems (§10.3): 1, 3, 5

BONUS NOTES

None.

REFERENCES

Zill, D. G., & Wright, W. S. (2014). *Advanced Engineering Mathematics* (5th ed.). Burlington, MA: Jones & Bartlett Learning.