Lecture VIII

Complex Numbers: Definitions, Powers & Roots

We are almost ready to approach higher-order ODEs, however we must first understand how to use complex numbers since they will be a necessity as you will soon find out. A complex number is any number of the form $z = \alpha + i\beta$, where α and β are real numbers and i is the imaginary unit defined by $i^2 = -1$. Here are some properties of complex numbers:

1) Equality: Two complex numbers $z_1 = \alpha_1 + i\beta_1$ and $z_2 = \alpha_2 + i\beta_2$ are equal ($z_1 = z_2$) if and only if $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$, that is to say $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$.
2) Addition: $z_1 + z_2 = (\alpha_1 + i\beta_1) + (\alpha_2 + i\beta_2) = (\alpha_1 + \alpha_2) + i(\beta_1 + \beta_2)$
3) Subtraction: $z_1 - z_2 = (\alpha_1 + i\beta_1) - (\alpha_2 + i\beta_2) = (\alpha_1 - \alpha_2) + i(\beta_1 - \beta_2)$
4) Multiplication: $z_1 z_2 = (\alpha_1 + i\beta_1)(\alpha_2 + i\beta_2) = (\alpha_1\alpha_2 - \beta_1\beta 2) + i(\alpha_1\beta_2 + \alpha_2\beta_1)$
5) Division: $\dfrac{z_1}{z_2} = \dfrac{\alpha_1 + i\beta_1}{\alpha_2 + i\beta_2} = \dfrac{\alpha_1\alpha_2 + \beta_1\beta_2}{\alpha_2^2 + \beta_2^2} + i\dfrac{\alpha_2\beta_1 - \alpha_1\beta_2}{\alpha_2^2 + \beta_2^2}$
6) Commutative Laws: $z_1 + z_2 = z_2 + z_1$ and $z_1 z_2 = z_2 z_1$
7) Associative Laws: $z_1(z_2 + z_3) = (z_1 + z_2) + z_3$ and $z_1(z_2 z_3) = (z_1 z_2) z_3$
8) Distributive Law: $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$
9) Given a complex number $z = \alpha + i\beta$, its complex conjugate is $\bar{z} = \alpha - i\beta$
10) Conjugate Properties: $\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$, $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$, and $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}$

The real and imaginary parts of a complex number z can be easily determined as

$$\mathrm{Re}(z) = \frac{z + \bar{z}}{2} \quad \text{and} \quad \mathrm{Im}(z) = \frac{z - \bar{z}}{2i} \tag{1}$$

and the modulus or absolute value of $z = \alpha + i\beta$ is defined by

$$|z| = \sqrt{\alpha^2 + \beta^2} = \sqrt{z\bar{z}}. \tag{2}$$

Complex numbers could be expressed in standard or rectangular form, $z = \alpha + i\beta$, however there are other forms which are of great use. One of which is the polar form $z = r(\cos(\theta) +$

$i\sin(\theta))$, where $r = |z|$ is the radius and $\theta = \arg(z)$ is called the argument, and represents the inclination measured counter-clockwise from the positive x-axis. The argument of a complex number expressed in the interval $-\pi < \theta \leq \pi$ is called the principal argument and is denoted by $\text{Arg}(z)$. One has to be careful when computing the principal argument of a complex number $z = \alpha + i\beta$, mainly a true understanding of the meaning of $\arctan(\frac{\beta}{\alpha})$ is needed. The number $z = \alpha + i\beta$ can be thought of as a point in the complex plane with α being the component on the real axis and β being the component on the imaginary axis. Depending on the quadrant of the complex plane we are in, we have to treat $\arctan(\frac{\beta}{\alpha})$ differently, i.e.

$$
\begin{aligned}
\text{Quadrant 1: } \alpha > 0 \ \&\ \beta > 0 \quad \rightarrow \quad 0 \leq & \ \arctan\left(\frac{\beta}{\alpha}\right) \leq \frac{\pi}{2} \\
\text{Quadrant 2: } \alpha < 0 \ \&\ \beta > 0 \quad \rightarrow \quad \frac{\pi}{2} \leq & \ \arctan\left(\frac{\beta}{\alpha}\right) \leq \pi \\
\text{Quadrant 3: } \alpha < 0 \ \&\ \beta < 0 \quad \rightarrow \quad -\pi \leq & \ \arctan\left(\frac{\beta}{\alpha}\right) \leq -\frac{\pi}{2} \\
\text{Quadrant 4: } \alpha > 0 \ \&\ \beta < 0 \quad \rightarrow \quad -\frac{\pi}{2} \leq & \ \arctan\left(\frac{\beta}{\alpha}\right) \leq 0.
\end{aligned}
$$

The rules of trigonometry make some operations on complex numbers expressed in polar form somewhat simple, for instance

$$
\begin{aligned}
z_1 z_2 &= r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] \\
\frac{z_1}{z_2} &= \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)] \\
|z_1 z_2| &= |z_1||z_2| \\
\left|\frac{z_1}{z_2}\right| &= \frac{|z_1|}{|z_2|} \\
\arg(z_1 z_2) &= \arg(z_1) + \arg(z_2) \\
\arg\left(\frac{z_1}{z_2}\right) &= \arg(z_1) - \arg(z_2).
\end{aligned}
$$

Particularly useful is the fact that

$$z^n = r^n(\cos(n\theta) + i\sin(n\theta)), \tag{3}$$

owing to DeMoivre's formula $(\cos(\theta)+i\sin(\theta))^n = \cos(n\theta)+i\sin(n\theta)$, which will enable us to determine the roots of complex numbers. A number $w = \rho(\cos(\phi) + i\sin(\phi))$ (where $\rho = |w|$ and $\phi = \text{Arg}(w)$) is said to be an nth root of a complex number $z = r(\cos(\theta) + i\sin(\theta))$ (where $r = |z|$ and $\theta = \text{Arg}(z)$) if $w^n = z$, namely

$$w^n = \rho^n(\cos(n\phi) + i\sin(n\phi)) = r(\cos(\theta) + i\sin(\theta)).$$

This implies that

$$\rho = r^{1/n}$$
$$n\phi = \theta + 2k\pi \quad \text{or} \quad \phi = \frac{\theta + 2k\pi}{n} \quad \text{for} \quad k = 0, 1, 2, ..., n-1,$$

therefore the n roots of a complex number must be given by

$$w_k = r^{1/n}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right]. \tag{4}$$

Another useful form to use for complex numbers is the exponential $z = re^{i\theta}$ owing to the identity $e^{i\theta} = \cos(\theta) + i\sin(\theta)$.

EXAMPLE

Find all the roots of $\sqrt[3]{1-\sqrt{3}i}$.

The complex number $z = 1 - \sqrt{3}i$ has $r = |z| = 4$ and $\theta = \text{Arg}(z) = -\pi/3$. Therefore $\rho = \sqrt[3]{4}$ and the three roots are

$$\begin{aligned}
k = 0 \quad &\rightarrow \quad w_0 = \sqrt[3]{4}\left[\cos\left(-\tfrac{\pi}{9}\right) + i\sin\left(-\tfrac{\pi}{9}\right)\right] \\
k = 1 \quad &\rightarrow \quad w_1 = \sqrt[3]{4}\left[\cos\left(\tfrac{5\pi}{9}\right) + i\sin\left(\tfrac{5\pi}{9}\right)\right] \\
k = 2 \quad &\rightarrow \quad w_2 = \sqrt[3]{4}\left[\cos\left(\tfrac{11\pi}{9}\right) + i\sin\left(\tfrac{11\pi}{9}\right)\right].
\end{aligned}$$

Lecture Problems (§17.1): 2, 20 & **(§17.2)** 9, 30

Tutorial Problems (§17.1): 5, 6, 19, 20 & **(§17.2)** 6, 10, 12, 32

Suggested Problems (§17.1): 1, 7, 13, 23, 33, 37 & **(§17.2)** 1, 7, 11, 17, 21

BONUS NOTES

None.

REFERENCES

Zill, D. G., & Wright, W. S. (2014). *Advanced Engineering Mathematics* (5th ed.). Burlington, MA: Jones & Bartlett Learning.