Lecture XVII

Review of Power Series

Before learning how to solve differential equations using power series solutions, a review of power series is in order. A power series in x centered at the point $x = a$ is an infinite series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots . \tag{1}$$

Below, a few facts about power series are summarized.

Convergence

A power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ is convergent at a specific value of x if its sequence of partial sums $\{S_N(x)\}$ converges; that is, if $\lim_{N\to\infty} \sum_{n=0}^{N} c_n (x-a)^n$ exists. If the limit does not exist at x, the series is said to be divergent.

Interval of Convergence

Every power series has an interval of convergence. The interval of convergence is the set of all real numbers x for which the series converges.

Radius of Convergence

Every power series has a radius of convergence R. If $R > 0$, then a power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ converges for $|x-a| < R$ and diverges for $|x-a| > R$. If the series converges only at its center a, then $R = 0$. If the series converges for all x, then we write $R = \infty$. Recall that the absolute-value inequality $|x-a| < R$ is equivalent to the simultaneous inequality $a - R < x < a + R$. A power series may or may not converge at the endpoints $a - R$ and $a + R$ of this interval.

Absolute Convergence

Within its interval of convergence a power series converges absolutely. In other words, if x is a number in the interval of convergence and is not an endpoint of the interval, then the series of absolute values $\sum_{n=0}^{\infty} |c_n (x-a)^n|$ converges.

Ratio Test

Convergence of power series can often be determined by the ratio test. Suppose that $c_n \neq 0$ for all n, and that

$$\lim_{n\to\infty} \left| \frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n} \right| = |x-a| \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| = L. \tag{2}$$

If $L < 1$ the series converges absolutely, if $L > 1$ the series diverges, and if $L = 1$ the test is inconclusive.

A Power Series Defines a Function

A power series defines a function $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ whose domain is the interval of convergence of the series. If the radius of convergence is $R > 0$, then f is continuous, differentiable, and integrable on the interval $(a-R, a+R)$. Moreover, $f'(x)$ and $\int f(x)\mathrm{d}x$ can be found by term-by-term differentiation and integration. Convergence at an endpoint may be either lost by differentiation or gained through integration. If $y = \sum_{n=0}^{\infty} c_n x^n$ is a power series in x, then the first two derivatives are $y' = \sum_{n=0}^{\infty} c_n n x^{n-1}$ and $y'' = \sum_{n=0}^{\infty} c_n n(n-1)x^{n-2}$. Notice that the first term in the first derivative and the first two terms in the second derivative are zero. We omit these zero terms and write

$$y' = \sum_{n=1}^{\infty} c_n n x^{n-1} \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2}.$$

Identity Property

If $\sum_{n=0}^{\infty} c_n(x-a)^n = 0$, $R > 0$, for all numbers x in the interval of convergence, then $c_n = 0$ for all n.

Analytic at a Point

A function f is analytic at a point a if it can be represented by a power series in $x - a$ with a positive radius of convergence.

A useful trick in adding or subtracting power series is the shifting of the summation index. We will be using this quite often in the next lecture.

EXAMPLE

$$\text{Write} \quad \sum_{n=2}^{\infty}(n-1)(n+1)c_n x^{n-2} + \sum_{n=1}^{\infty} nc_n x^n \quad \text{as one power series.}$$

To group the power series, we need to ensure that both the summation indices start at the same number and that the powers of x in each series be "in phase." Let's start by equating the powers. The first series starts with x^0 while the second starts with x^1, so we should first pull the first term out of the first series

$$3c_2 + \sum_{n=3}^{\infty}(n-1)(n+1)c_n x^{n-2} + \sum_{n=1}^{\infty} nc_n x^n.$$

Now that they start at the same power, we could relabel the indices so that they both start at the same value. In the first power series, we could say $k = n - 2$ giving

$$3c_2 + \sum_{k=1}^{\infty}(k+1)(k+3)c_{k+2} x^k + \sum_{n=1}^{\infty} nc_n x^n.$$

In the second series, we could say $k = n$, giving

$$3c_2 + \sum_{k=1}^{\infty}(k+1)(k+3)c_{k+2} x^k + \sum_{k=1}^{\infty} kc_k x^k.$$

We can now sum the two series to give the final result

$$3c_2 + \sum_{k=1}^{\infty}\left[kc_k + (k+1)(k+3)c_{k+2}\right] x^k.$$

Lecture Problems (§5.1): 2, 11

Tutorial Problems (§5.1): 4, 12

Suggested Problems (§5.1): 1, 3

BONUS NOTES

None.

REFERENCES

Zill, D. G., & Wright, W. S. (2014). *Advanced Engineering Mathematics* (5th ed.). Burlington, MA: Jones & Bartlett Learning.