Lecture XXII

Non-Homogeneous Linear Systems

In solving the nonhomogeneous linear system of n ODEs $\mathbf{X}' = \mathbf{AX} + \mathbf{F}$, we essentially have the same methods at our disposal as previously learned, namely undetermined coefficients, variation of parameters, and diagonalization.

Undetermined Coefficients

For undetermined coefficients, the method essentially proceeds as we would expect, namely by assuming the form of each x in $\mathbf{X}$. This is best illustrated through an example.

EXAMPLE (from Zill & Wright, 2014)

$$\text{Solve} \quad \mathbf{X}' = \begin{bmatrix} 6 & 1 \\ 4 & 3 \end{bmatrix} \mathbf{X} + \begin{bmatrix} 6t \\ -10t + 4 \end{bmatrix}$$

We must first get the homogeneous solution. In this case we have two distinct eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 7$, so the corresponding homogeneous solution is

$$\mathbf{X}_h = c_1 \begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{7t}$$

Our assumed form $\mathbf{X}_p$ is

$$\mathbf{X}_p = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} t + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

so upon substituting it in we get

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 6 & 1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} a_1 t + b_1 \\ a_2 t + b_2 \end{bmatrix} + \begin{bmatrix} 6t \\ -10t + 4 \end{bmatrix}.$$

Comparing both sides of the equation, we get four equations in four unknowns, namely

$$6a_2 + b_2 + 6 = 0 \quad 4a_2 + 3b_2 - 10 = 0$$
$$6a_1 + b_1 - a_2 = 0 \quad 4a_1 + 3b_1 - b_2 + 4 = 0$$

with solution $a_1 = -4/7$, $b_1 = 10/7$, $a_2 = -2$, and $b_2 = 6$. The general solution is therefore

$$\mathbf{X} = \begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{7t} + \begin{bmatrix} -4/7 \\ -2 \end{bmatrix} t + \begin{bmatrix} 10/7 \\ 6 \end{bmatrix}.$$

We must note however that the method of undetermined coefficients isn't as straightforward for linear systems. Namely, since it is a system of equations, we have to use the **same** form for each row of $\mathbf{X}_p$. For instance, if our equation were

$$\mathbf{X}' = \begin{bmatrix} 5 & 3 \\ -1 & 1 \end{bmatrix} \mathbf{X} + \begin{bmatrix} -2 \\ 0 \end{bmatrix} e^{2t} + \begin{bmatrix} 0 \\ 5 \end{bmatrix} t + \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

whose homogeneous solution is

$$\mathbf{X}_h = c_1 \begin{bmatrix} -3 \\ 1 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{2t},$$

we would assume

$$\mathbf{X}_p = \begin{bmatrix} a_4 \\ b_4 \end{bmatrix} te^{2t} + \begin{bmatrix} a_3 \\ b_3 \end{bmatrix} e^{2t} + \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} t + \begin{bmatrix} a_1 \\ b_1 \end{bmatrix}.$$

Note that we explicitly put $a_2 \neq 0$ even though there is no "t" contribution in the first row of F and that we put $b_4 \neq 0$ and $b_3 \neq 0$ even though there is no "e^{2t}" contribution in the second row of F.

Variation of Parameters

Let us define the **fundamental matrix** of the system $\mathbf{X}' = \mathbf{A}\mathbf{X}$ as

$$\mathbf{\Phi}(t) = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{bmatrix} \tag{1}$$

such that the solution to the system $\mathbf{X}' = \mathbf{A}\mathbf{X}$ could be expressed as $\mathbf{X} = \mathbf{\Phi}(t)\mathbf{C}$. Note also that $\mathbf{\Phi}'(t) = \mathbf{A}\mathbf{\Phi}(t)$. As before, knowing the homogeneous solution (which is contained in $\mathbf{\Phi}$), let us search for the particular solution of the system $\mathbf{X}' = \mathbf{A}\mathbf{X} + \mathbf{F}$ as

$$\mathbf{X}_p = \mathbf{\Phi}(t)\mathbf{U}(t) \quad \text{where} \quad \mathbf{U}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_n(t) \end{bmatrix}.$$

Substituting this into the equation yields the following

$$\begin{aligned} \mathbf{X}' &= \mathbf{A}\mathbf{X} + \mathbf{F}(t) \\ \mathbf{\Phi}(t)\mathbf{U}'(t) + \mathbf{\Phi}'(t)\mathbf{U}(t) &= \mathbf{A}\mathbf{\Phi}(t)\mathbf{U}(t) + \mathbf{F}(t) \\ \mathbf{\Phi}(t)\mathbf{U}'(t) &= \mathbf{F}(t) \\ \mathbf{U}'(t) &= \mathbf{\Phi}^{-1}(t)\mathbf{F}(t) \\ \mathbf{U}(t) &= \int \mathbf{\Phi}^{-1}(t)\mathbf{F}(t)\mathrm{d}t \\ \mathbf{X}_p = \mathbf{\Phi}(t)\mathbf{U}(t) &= \mathbf{\Phi}(t)\int \mathbf{\Phi}^{-1}(t)\mathbf{F}(t)\mathrm{d}t. \end{aligned}$$

The general solution is therefore

$$\mathbf{X} = \mathbf{\Phi}(t)\mathbf{C} + \mathbf{\Phi}(t)\int \mathbf{\Phi}^{-1}(t)\mathbf{F}(t)\mathrm{d}t. \tag{2}$$

Consequently, for the initial value problem $\mathbf{X}' = \mathbf{A}\mathbf{X} + \mathbf{F}(t)$ subject to $\mathbf{X}(t_0) = \mathbf{X}_0$, the general solution is

$$\mathbf{X} = \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(t_0)\mathbf{X}_0 + \mathbf{\Phi}(t)\int_{t_0}^{t} \mathbf{\Phi}^{-1}(s)\mathbf{F}(s)\mathrm{d}s. \tag{3}$$

Diagonalization

Given the system $\mathbf{X}' = \mathbf{AX} + \mathbf{F}(t)$, suppose $\mathbf{P}$ is the matrix such that $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$, where $\mathbf{D}$ is a diagonal matrix. Substituting $\mathbf{X} = \mathbf{PY}$ gives

$$\begin{aligned} \mathbf{PY}' &= \mathbf{APY} + \mathbf{F} \\ \mathbf{Y}' &= \mathbf{P}^{-1}\mathbf{APY} + \mathbf{P}^{-1}\mathbf{F} \\ \mathbf{Y}' &= \mathbf{DY} + \mathbf{G}. \end{aligned}$$

Each differential equation in this new system is decoupled. Once we have solved for $\mathbf{Y}$ (with $\mathbf{G} = \mathbf{P}^{-1}\mathbf{F}$), then we can solve for $\mathbf{X}$ from $\mathbf{X} = \mathbf{P}^{-1}\mathbf{Y}$.

Lecture Problems (§10.4): 2, 16

Tutorial Problems (§10.4): 4, 6, 14, 18

Suggested Problems (§10.4): 1, 3, 5, 9, 13, 15, 19

BONUS NOTES

None.

REFERENCES

Zill, D. G., & Wright, W. S. (2014). *Advanced Engineering Mathematics* (5th ed.). Burlington, MA: Jones & Bartlett Learning.