Course Summary

Steps to First-Order ODE Happiness

Step 1: ***Is this a first-order separable equation?***

Can the equation be expressed in the form below?

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x,y) \quad \text{where} \quad f(x,y) = g(x)h(y)$$

1) Put all the y's and dy's on the left-hand side of the equation.

2) Put all the x's and dx's on the right-hand side of the equation.

3) Integrate both sides (don't forget the constant of integration on the right-hand side).

4) Solve for y (if possible) and simplify the result.

Step 2: ***Is this a linear first-order equation?***

Can the equation be expressed in the form below?

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)$$

If not, does the equation become linear when we interchange the dependent and independent variables?

1) Compute the integrating factor.

$$I(x) = e^{\int P(x)\mathrm{d}x}$$

2) Substitute $I(x)$ into the simplified form of the equation.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[I(x)y\right] = I(x)Q(x)$$

3) Integrate both sides (don't forget the constant of integration on the right-hand side).

4) Solve for y and simplify the result.

Step 3: ***Is this an exact equation?***

Can the equation be expressed in the form below?

$$M(x,y)\mathrm{d}x + N(x,y)\mathrm{d}y = 0 \quad \text{where} \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If not, can I make the equation exact after multiplying through by an integrating factor (μ)?

$$\mu(x) = e^{\int \frac{M_y - N_x}{N}\mathrm{d}x} \quad \text{iff} \quad \frac{M_y - N_x}{N} \neq f(y) \quad \text{or} \quad \mu(y) = e^{\int \frac{N_x - M_y}{M}\mathrm{d}y} \quad \text{iff} \quad \frac{N_x - M_y}{M} \neq f(x).$$

1) Compute the solution $f(x,y)$ by integrating $M(x,y)$ with respect to x.

$$\frac{\partial f}{\partial x} = M(x,y) \to f(x,y) = \int M(x,y)\mathrm{d}x + g(y)$$

2) Compute the partial derivative of the $f(x,y)$ obtained in step (1) with respect to y and determine $g(y)$ using $N(x,y)$.

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\int M(x,y)\mathrm{d}x + \frac{\mathrm{d}g}{\mathrm{d}y} = N(x,y) \to \text{Solve for } g(y).$$

3) Assemble the solution (replace the $g(y)$ in step (1) with what was found in step (2)).

$$f(x,y) = \int M(x,y)\mathrm{d}x + g(y) + c = 0 \quad \text{or} \quad \int M(x,y)\mathrm{d}x + g(y) = c$$

4) Solve for y (if possible) and simplify the result.

Step 4: ***Is this equation composed of homogeneous functions of the same degree?***

Can the equation be expressed in the form below?

$$M(x,y)\mathrm{d}x + N(x,y)\mathrm{d}y = 0$$

where $M(x,y)$ and $N(x,y)$ are homogeneous functions of the same degree; recall that a homogeneous function possesses the property $f(tx,ty) = t^{\alpha} f(x,y)$.

1) Use one of the substitutions below.

$$\text{a)} \quad y = ux \quad \text{and} \quad \mathrm{d}y = u\mathrm{d}x + x\mathrm{d}u$$

$$\text{b)} \quad x = vy \quad \text{and} \quad \mathrm{d}x = v\mathrm{d}y + y\mathrm{d}v.$$

2) The equation is now a separable equation, so solve for u or v.

3) Substitute back $u = y/x$ or $v = x/y$.

4) Solve for y (if possible) and simplify the result.

Step 5: ***Is this a Bernoulli equation?***

Can the equation be expressed in the form below?

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)y^n$$

If not, does the equation become a Bernoulli equation when we interchange the dependent and independent variables?

1) Determine the value of n.

2) Using the substitution $u = y^{1-n}$, we obtain the linear equation:

$$\frac{\mathrm{d}u}{\mathrm{d}x} + (1-n)P(x)u = (1-n)Q(x)$$

3) Solve for u and substitute back $u = y^{1-n}$.

4) Solve for y and simplify the result.

REFERENCES

Zill, D. G., & Wright, W. S. (2014). *Advanced Engineering Mathematics* (5th ed.). Burlington, MA: Jones & Bartlett Learning.