Lecture XXIV

The Matrix Exponential

A rather simple way of expressing the solution of the homogeneous linear system $\mathbf{X}' = \mathbf{A}\mathbf{X}$, is using the matrix exponential,

$$\mathbf{X} = e^{\mathbf{A}t}\mathbf{C}, \tag{1}$$

where the column vector $\mathbf{C}$ contains the arbitrary constants. Furthermore, the general solution of the nonhomogeneous system $\mathbf{X}' = \mathbf{A}\mathbf{X} + \mathbf{F}(t)$ is

$$\mathbf{X} = e^{\mathbf{A}t}\mathbf{C} + e^{\mathbf{A}t}\int_{t_0}^{t} e^{\mathbf{A}s}\mathbf{F}(s)\mathrm{d}s. \tag{2}$$

DEFINITION: Matrix Exponential

"For any $n \times n$ matrix $\mathbf{A}$,

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \mathbf{A}^2\frac{t^2}{2!} + \cdots + \mathbf{A}^m\frac{t^m}{m!} + \cdots = \sum_{k=0}^{\infty}\mathbf{A}^k\frac{t^k}{k!}$$

(Zill & Wright, 2014)."

Quite naturally, the derivative of $e^{\mathbf{A}t}$ is simply $\mathbf{A}e^{\mathbf{A}t}$. Can we compute the matrix exponential without resorting to its definition? There are two common ways. The first is using Laplace transforms which you will learn in ENGR 311. The second is using the powers of $\mathbf{A}^m$, since we can write

$$\mathbf{A}^m = \sum_{j=0}^{m-1} c_j\mathbf{A}^j,$$

therefore

$$e^{\mathbf{A}t} = \sum_{k=0}^{\infty}\mathbf{A}^k\frac{t^k}{k!} = \sum_{k=0}^{\infty}\frac{t^k}{k!}\left(\sum_{j=0}^{m-1}c_j(k)\mathbf{A}^j\right) = \sum_{j=0}^{m-1}\mathbf{A}^j\underbrace{\left(\sum_{k=0}^{\infty}\frac{t^k}{k!}c_j(k)\right)}_{b_j(t)} = \sum_{j=0}^{m-1}\mathbf{A}^j b_j(t).$$

Lecture Problems (§10.5):

Tutorial Problems (§10.5):

Suggested Problems (§10.5):

BONUS NOTES

None.

REFERENCES

Zill, D. G., & Wright, W. S. (2014). *Advanced Engineering Mathematics* (5th ed.). Burlington, MA: Jones & Bartlett Learning.