Lecture X

Reduction of Order

Given a homogeneous linear second-order differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0, \tag{1}$$

we now know that the solution is a linear combination of two solutions $y = c_1y_1 + c_2y_2$, where y_1 and y_2 constitute a linearly independent set on some interval I. It turns out that if we know one of the solutions to the above ODE, say $y_1(x)$, then we can reduce the ODE to a first-order ODE by means of a substitution in order to find the second solution $y_2(x)$. The method proceeds as follows, we will first put the equation in standard form

$$y'' + P(x)y' + Q(x)y = 0. \tag{2}$$

Now we suppose we have the solution $y_1(x)$. If y_1 and y_2 are linearly independent, then their ratio y_2/y_1 cannot be a constant, in other words it must be some function of x, i.e. $y_2/y_1 = u(x)$ or $y_2 = u(x)y_1$. Using this substitution then, we have

$$\begin{gathered}
y_2'' + P(x)y_2' + Q(x)y_2 = 0 \\
(uy_1)'' + P(x)(uy_1)' + Q(x)(uy_1) = 0 \\
(u''y_1 + 2u'y_1' + uy_1'') + P(x)(u'y_1 + uy_1') + Q(x)(uy_1) = 0 \\
u(\underbrace{y_1'' + P(x)y_1' + Q(x)y_1}_{= 0, \text{ since } y_1 \text{ is a solution}}) + y_1u'' + (2y_1' + P(x)y_1)u' = 0.
\end{gathered}$$

Substituting $w = u'$ and $w' = u''$,

$$y_1w' + (2y_1' + P(x)y_1)w = 0$$

we have a separable equation with solution

$$\frac{\mathrm{d}w}{w} + \left(2\frac{y_1'}{y_1} + P(x)\right)\mathrm{d}x = 0 \xrightarrow{\text{integrate}} \ln|w| + \ln\left|y_1^2\right| + \int P(x)\mathrm{d}x = c$$

$$w = k_1 \frac{e^{-\int P(x)\mathrm{d}x}}{y_1^2} \quad (\text{where } k_1 = e^c)$$

$$u(x) = k_1 \int \frac{e^{-\int P(x)\mathrm{d}x}}{y_1^2}\mathrm{d}x + k_2.$$

Choosing $k_1 = 1$ and $k_2 = 0$ (since they can be incorporated in the complete solution $y = c_1 y_1 + c_2 y_2$ anyway), the second solution can be expressed as

$$y_2(x) = y_1(x) \int \frac{e^{-\int P(x)\mathrm{d}x}}{y_1^2}\mathrm{d}x. \tag{3}$$

EXAMPLE

Determine the general solution of $2y'' - 8y' + 8y = 0$ knowing that one solution is $y_1 = e^{2x}$. First we put the equation in standard form

$$y'' - 4y' + 4y = 0,$$

then we can solve for y_2

$$y_2(x) = e^{2x} \int \frac{e^{\int 4\mathrm{d}x}}{e^{4x}}\mathrm{d}x$$

$$y_2(x) = xe^{2x}.$$

The general solution then is

$$y = c_1 y_1 + c_2 y_2 = c_1 e^{2x} + c_2 x e^{2x}.$$

EXAMPLE

Determine the general solution of $12x^2 y'' + 24xy' + 3y = 0$ knowing that one solution is

$y_1 = x^{-1/2}$. First we put the equation in standard form

$$y'' + \frac{2}{x}y' + \frac{1}{4}\frac{1}{x^2}y = 0,$$

then we can solve for y_2

$$y_2(x) = x^{-1/2} \int \frac{e^{-\int \frac{2}{x}\mathrm{d}x}}{\left(x^{-1/2}\right)^2}\mathrm{d}x$$

$$y_2(x) = x^{-1/2}\ln|x|.$$

The general solution then is

$$y = c_1y_1 + c_2y_2 = c_1x^{-1/2} + c_2x^{-1/2}\ln|x|.$$

Lecture Problems (§3.2): 2, 12

Tutorial Problems (§3.2): 4, 8, 10, 16

Suggested Problems (§3.2): 1, 3, 11, 17, 19

BONUS NOTES

We may also note that we could have determined y_2 using an integrating factor since the equation in w is both separable and linear, i.e.

$$\begin{aligned}
y_1 w' + (2y_1' + P(x)y_1)w &= 0 \\
w' + \left(2\frac{y_1'}{y_1} + P(x)\right) w &= 0 \\
I(x) = e^{\int\left(2\frac{y_1'}{y_1}+P(x)\right)\mathrm{d}x} = e^{\ln\left|y_1^2\right| + \int P(x)\mathrm{d}x} &= y_1^2 e^{\int P(x)\mathrm{d}x} \\
\frac{\mathrm{d}}{\mathrm{d}x}\left[y_1^2 e^{\int P(x)\mathrm{d}x} w\right] &= 0 \\
y_1^2 e^{\int P(x)\mathrm{d}x} w &= k_1 \\
w &= k_1 \frac{e^{-\int P(x)\mathrm{d}x}}{y_1^2} \\
u(x) &= k_1 \int \frac{e^{-\int P(x)\mathrm{d}x}}{y_1^2}\mathrm{d}x + k_2 \\
y_2(x) &= y_1(x) \int \frac{e^{-\int P(x)\mathrm{d}x}}{y_1^2}\mathrm{d}x
\end{aligned}$$

where we have set $k_1 = 1$ and $k_2 = 0$.

There is also another interesting way we can use the reduction of order method. Say we have the nonhomogeneous equation

$$y'' + P(x)y' + Q(x)y = g(x)$$

where one solution of the associated homogeneous equation $y_1(x)$ is known. If we use the same substitution uy_1 in the full nonhomogeneous equation, the particular solution can be obtained as well (in fact, what we are truly doing is using the substitution $y_2(x) + y_p(x) = uy_1$).

$$\begin{aligned}
y_1 w' + (2y_1' + P(x)y_1)w &= g(x) \\
w' + \left(2\frac{y_1'}{y_1} + P(x)\right) w &= \frac{g(x)}{y_1} \\
I(x) = e^{\int\left(2\frac{y_1'}{y_1}+P(x)\right)\mathrm{d}x} = e^{\ln\left|y_1^2\right| + \int P(x)\mathrm{d}x} &= y_1^2 e^{\int P(x)\mathrm{d}x}
\end{aligned}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[y_1^2 e^{\int P(x)\mathrm{d}x} w\right] = y_1 e^{\int P(x)\mathrm{d}x} g(x)$$

$$y_1^2 e^{\int P(x)\mathrm{d}x} w = \int y_1 e^{\int P(x)\mathrm{d}x} g(x)\mathrm{d}x + k_1$$

$$w = \frac{\int y_1 e^{\int P(x)\mathrm{d}x} g(x)\mathrm{d}x}{y_1^2 e^{\int P(x)\mathrm{d}x}} + k_1 \frac{e^{-\int P(x)\mathrm{d}x}}{y_1^2}$$

$$u(x) = \int \frac{\int y_1 e^{\int P(x)\mathrm{d}x} g(x)\mathrm{d}x}{y_1^2 e^{\int P(x)\mathrm{d}x}} \mathrm{d}x + k_1 \int \frac{e^{-\int P(x)\mathrm{d}x}}{y_1^2} \mathrm{d}x + k_2$$

$$y_p(x) = y_1 \int \frac{\int y_1 e^{\int P(x)\mathrm{d}x} g(x)\mathrm{d}x}{y_1^2 e^{\int P(x)\mathrm{d}x}} \mathrm{d}x \quad \text{and} \quad y_2(x) = y_1(x) \int \frac{e^{-\int P(x)\mathrm{d}x}}{y_1^2} \mathrm{d}x$$

where, as before, we have set $k_1 = 1$ and $k_2 = 0$.

REFERENCES

Zill, D. G., & Wright, W. S. (2014). *Advanced Engineering Mathematics* (5th ed.). Burlington, MA: Jones & Bartlett Learning.