Lecture IX

Theory of Linear Equations

Now that we have learned how to solve first-order ODEs, we will begin to approach higher-order ODEs. In particular, we will look at linear nth-order ODEs of the form

$$a_n(x)\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + a_{n-1}(x)\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \cdots + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = g(x). \tag{1}$$

We can also define an initial value problem as

$$\begin{gathered} a_n(x)\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + a_{n-1}(x)\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \cdots + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = g(x) \\ \text{subject to} \\ y(x_0) = y_0, y'(x_0) = y_0', y''(x_0) = y_0'', ..., y^{(n-1)}(x_0) = y_0^{(n-1)}. \end{gathered} \tag{2}$$

As for the first-order linear ODE, there is an existence and uniqueness theorem for such an initial value problem.

THEOREM: Existence of a Unique Solution

"Let $a_n(x), a_{(n-1)}(x), ..., a_1(x), a_0(x)$, and $g(x)$ be continuous on an interval I, and let $a_n(x) \neq 0$ for every x in this interval. If $x = x_0$ is any point in this interval, then a solution $y(x)$ of the initial value problem (2) exists on the interval and is unique." (Zill & Wright, 2014)

We should note that for an *nth* order ODE, there is another type of problem that can be described. In an **initial value problem**, the values for $y, y', ..., y^{(n-1)}$ are prescribed at a single point x_0. In the case of a **boundary value problem**, we have conditions prescribed which may be at multiple points and may also be a linear combination of $y, y', ..., y^{(n-1)}$. For instance, for a second-order ODE we could have the following boundary conditions rather than the initial conditions $y(x_0) = y_0$ and $y'(x_0) = y_0'$,

$$A_1 y(a) + B_1 y'(a) = C_1$$

$$A_2 y(b) + B_2 y'(b) = C_2.$$

In general, a boundary value problem may have a unique solution, many solutions, or no solution.

A **homogeneous equation** is one for which the equation is set equal to zero, i.e.

$$a_n(x)\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + a_{n-1}(x)\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \cdots + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = 0, \tag{3}$$

while a nonhomogeneous equation is nonzero (equal to some function $g(x) \neq 0$),

$$a_n(x)\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + a_{n-1}(x)\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \cdots + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = g(x). \tag{4}$$

You will see that in order to solve a nonhomogeneous ODE, we need to include the solution to the associated homogeneous ODE, since the contribution of the solution to the associated homogeneous ODE amounts to zero.

Please find below several definitions and theorems for linear nth-order ODEs.

THEOREM: Superposition Principle for Homogeneous Equations

"Let $y_1, y_2, ..., y_k$ be solutions of the homogeneous nth-order ODE (3) on an interval I. Then the linear combination

$$y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_k y_k(x),$$

where the c_j, $j = 1, 2, ..., k$ are arbitrary constants, is also a solution on the interval." (Zill & Wright, 2014)

COROLLARIES

1) "A constant multiple $y = c_1 y_1(x)$ of a solution $y_1(x)$ of a homogeneous linear ODE is also a solution." (Zill & Wright, 2014)

2) "A homogeneous linear ODE always possesses the trivial solution $y = 0$." (Zill & Wright, 2014)

DEFINITION: Linear Dependence & Independence

"A set of functions $f_1(x), f_2(x), ..., f_n(x)$ is said to be **linearly dependent** on an interval I if there exists constants $c_1, c_2, ..., c_n$, not all zero, such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$$

for every x in the interval. If the set of functions is not linearly dependent on the interval, it is said to be **linearly independent**." (Zill & Wright, 2014)

In the case of two functions, this is rather easy to understand. If two functions $f(x)$ and $g(x)$ are linearly dependent, then one function must be a constant multiple of the other (i.e. $f(x) = cg(x)$). If two functions $f(x)$ and $g(x)$ are linearly independent, then the functions cannot be constant multiples of each other, but rather is some other function of x (i.e. $f(x)/g(x) = u(x)$).

DEFINITION: Wronskian

"Suppose each of the functions $f_1(x), f_2(x), ..., f_n(x)$ possesses at least $n-1$ derivatives. The determinant

$$W(f_1, f_2, ..., f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}.$$

where the primes denote derivatives, is called the **Wronskian** of the functions." (Zill & Wright, 2014)

THEOREM: Criterion for Linearly Independent Solutions

"Let $y_1, y_2, ..., y_n$ be n solutions of the homogeneous linear nth-order ODE (3) on an interval I. Then the set of solutions is **linearly independent** on I if and only if their $W(y_1, y_2, ..., y_n) \neq 0$ for every x in I." (Zill & Wright, 2014)

DEFINITION: Fundamental Set of Solutions

"Any set $y_1, y_2, ..., y_n$ of n linearly independent solutions of the homogeneous linear nth-order ODE (3) on an interval I is said to be a **fundamental set of solutions** on the interval." (Zill & Wright, 2014)

THEOREM: Existence of a Fundamental Set

"There exists a fundamental set of solutions for the homogeneous linear nth-order ODE (3) on an interval I." (Zill & Wright, 2014)

THEOREM: General Solution of Homogeneous Equations "Let $y_1, y_2, ..., y_n$ be a fundamental set of solutions of the homogeneous linear nth-order ODE (3) on an interval I. Then the **general solution** of the equation on the interval is

$$y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x),$$

where c_j, $j = 1, 2, ..., n$ are arbitrary constants." (Zill & Wright, 2014)

THEOREM: General Solution of Nonhomogeneous Equations "Let y_p be any particular solution of the nonhomogeneous linear nth-order ODE (4) on an interval I, and let $y_1, y_2, ..., y_n$ be a fundamental set of solutions of the associated homogeneous ODE (3) on I. Then the **general solution** of the equation on the interval is

$$y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) + y_p,$$

where the c_j, $j = 1, 2, ..., n$ are arbitrary constants." (Zill & Wright, 2014)

THEOREM: Superposition Principle for Nonhomogeneous Equations

"Let $y_{p_1}, y_{p_2}, ..., y_{p_k}$ be k particular solutions of the nonhomogeneous linear nth-order differential equation ODE (4) on an interval I corresponding, in turn, to k distinct functions $g_1, g_2, ..., g_k$. That is, suppose y_{p_j} denotes a particular solution of the corresponding differential equation

$$a_n(x)y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = g_j(x),$$

where $j = 1, 2, ..., k$. Then

$$y_p = y_{p_1}(x) + y_{p_2}(x) + \cdots + y_{p_k}(x)$$

is a particular solution of

$$a_n(x)y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = g_1(x) + g_2(x) + \cdots + g_k(x).\text{''}$$

(Zill & Wright, 2014)

Lecture Problems (§3.1): 16, 31

Tutorial Problems (§3.1): 3, 10, 17, 25, 33

Suggested Problems (§3.1): 1, 13, 15, 23

BONUS NOTES

None.

REFERENCES

Zill, D. G., & Wright, W. S. (2014). *Advanced Engineering Mathematics* (5th ed.). Burlington, MA: Jones & Bartlett Learning.