Lecture XVI

Linear Models: Initial & Boundary Value Problems

As we did for first-order ODEs, we will now look at some applications of higher-order linear ODEs in science.

Mass-Spring-Damper System

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + c\frac{\mathrm{d}x}{\mathrm{d}t} + kx = F(t) \quad \text{or} \quad \frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2\zeta\frac{\mathrm{d}x}{\mathrm{d}t} + \omega^2 x = F(t) \tag{1}$$

This equation results from the application of Newton's second law to a mass-spring-damper system with mass m, spring constant k, and damping constant c. It expresses the displacement x of the mass with time t. If the forcing function $F(x) = 0$ we have **free motion** which implies that motion can only arise from some initial condition, while if it is nonzero we have **driven motion** (i.e. the force is driving the motion of the system). If the forcing function and damping constant are zero, we have **free undamped motion**, implying that no energy is being dissipated in the system and so it will oscillate forever. The latter form of the equation is commonly used when there is damping in the system since it explicitly shows the natural frequency of the system $\omega = \sqrt{k/m}$ and the damping factor $2\zeta = c/m$. If we consider the case when the forcing function is zero, we have a linear second-order ODE with constant coefficients giving the three distinct cases: 1) **Overdamped Motion** when $\zeta^2 - \omega^2 > 0$ (distinct roots), 2) **Critically Damped Motion** when $\zeta^2 - \omega^2 = 0$ (repeated roots), and **Underdamped Motion** when $\zeta^2 - \omega^2 < 0$ (complex conjugate roots). **Conditions for a mass-spring-damper system are typically expressed as initial conditions, i.e.** $x(t_0) = x_0$ **and** $x'(t_0) = x'_0$.

RLC Circuits

$$L\frac{\mathrm{d}^2q}{\mathrm{d}t^2} + R\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{1}{C}q = E(t) \tag{2}$$

This equation results from the application of Kirchhoff's second law on a series circuit with a voltage source $E(t)$, a resistor R, an inductor L, and a capacitor C. It expresses the behaviour of the charge q with time t. Note that the equation looks very similar to the mass-spring-damper system, in fact they can be used as analogs of each other. As for the mass-spring-damper system, in the homogeneous case we have: 1) **Overdamped** when $R^2 - 4L/C > 0$ (distinct roots), 2) **Critically Damped** when $R^2 - 4L/C = 0$ (repeated roots), and **Underdamped** when $R^2 - 4L/C < 0$ (complex conjugate roots). **Conditions for an RLC circuit are typically expressed as initial conditions, i.e.** $q(t_0) = q_0$ **and** $q'(t_0) = i_0$.

Deflection of a Beam

$$EI\frac{\mathrm{d}^4y}{\mathrm{d}x^4} = w(x) \tag{3}$$

This equation expresses the deflection y of a beam with distance x, where EI is the flexural rigidity of the beam and $w(x)$ is the load distribution per unit length. **Conditions for a deflecting beam are typically expressed as boundary conditions.** At an embedded end of a beam (x_{EE}), we have $y(x_{EE}) = 0$ and $y'(x_{EE}) = 0$. At a free end of a beam (x_{FE}), we have $y''(x_{FE}) = 0$ and $y'''(x_{FE}) = 0$. If an end is simply supported (x_{SS}), we have $y(x_{SS}) = 0$ and $y''(x_{SS}) = 0$.

Buckling of a Vertical Column

$$EI\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + Py = 0 \tag{4}$$

This equation expresses the deflection y with distance x of a buckling vertical column with flexural rigidity EI and point load P. Typically we speak of the column as being pinned at both ends, giving us boundary conditions $y(0) = 0$ and $y(L) = 0$.

Rotating String

$$T\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \rho\omega^2 y = 0 \tag{5}$$

This equation expresses the deflection y with distance x of a string being rotated (like a jump rope) with tension T, mass per unit length ρ, and angular speed ω. Again, this type of problem is typically accompanied by boundary conditions where the ends are anchored, i.e. $y(0) = 0$ and $y(L) = 0$.

Lecture Problems (§3.8): 8 & **(§3.9):** 4a, 4b

Tutorial Problems (§3.8): 2, 4, 10 & **(§3.9):** 1, 2

Suggested Problems (§3.8): 3, 5, 9 & **(§3.9):** 3, 5a, 5b

BONUS NOTES

None.

REFERENCES

Zill, D. G., & Wright, W. S. (2014). *Advanced Engineering Mathematics* (5th ed.). Burlington, MA: Jones & Bartlett Learning.