Course Summary

Steps to Linear ODE Happiness

Step 1: ***Given one homogeneous solution, can I find a second one?***

Express the equation in the form below.

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + P(x)\frac{\mathrm{d}y}{\mathrm{d}x} + Q(x)y = 0$$

1) Establish your $P(x)$ from the equation.

2) Using the known homogeneous solution y_1, use the formula below to find the second one (ignore any constants of integration).

$$y_2(x) = y_1(x)\int \frac{e^{-\int P(x)\mathrm{d}x}}{y_1^2}\mathrm{d}x$$

3) Assemble the general homogeneous solution: $y_h = c_1y_1(x) + c_2y_2(x)$.

Step 2: ***Is this a linear homogeneous equation with constant coefficients?***

Express the equation in the form below.

$$a_2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + a_1\frac{\mathrm{d}y}{\mathrm{d}x} + a_0y = 0$$

1) Using $y = e^{mx}$, establish the characteristic polynomial.

$$a_2m^2 + a_1m + a_0 = 0$$

2) Determine the roots of the characteristic polynomial m_1 and m_2.

3) Establish the homogeneous solution based on the following three cases:

a) Distinct Roots (m_1 and m_2): $y_h = c_1e^{m_1x} + c_2e^{m_2x}$

b) Double Root ($m = m_1 = m_2$): $y_h = c_1e^{mx} + c_2xe^{mx}$

c) Complex Conjugate Roots ($m_{1,2} = \alpha \pm \beta i$): $y_h = e^{\alpha x}\left[c_1\cos(\beta x) + c_2\sin(\beta x)\right]$

* Know how this method generalizes to higher-order linear homogeneous ODEs with constant

coefficients.

Step 3: (Undetermined Coefficients) ***Is this a linear nonhomogeneous equation with constant coefficients?***

Express the equation in the form below.

$$a_2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + a_1 \frac{\mathrm{d}y}{\mathrm{d}x} + a_0 y = g(x)$$

1) Use the method of ***step 2*** to determine the homogeneous solution y_h (i.e. ignore $g(x)$).

2) Assume the form of the particular solution y_p based on the form of $g(x)$ using the table below. Note that if a "term" (y_{p_j}) in the assumed form of y_p exists in y_h, the "term" must be multiplied by x (i.e. $y_{p_j} \to x y_{p_j}$).

$g(x)$	Form of y_p
1. $c_n x^n + c_{n-1} x^{n-1} + \cdots + c_2 x^2 + c_1 x + c_0$	$A_n x^n + A_{n-1} x^{n-1} + \cdots + A_2 x^2 + A_1 x + A_0$
2. $c \cos(\alpha x)$	$A \cos(\alpha x) + B \sin(\alpha x)$
3. $c \sin(\alpha x)$	$A \cos(\alpha x) + B \sin(\alpha x)$
4. $c e^{\alpha x}$	$A e^{\alpha x}$
5. $c e^{\alpha x} \cos(\beta x)$	$A e^{\alpha x} \cos(\beta x) + B e^{\alpha x} \sin(\beta x)$
6. $c e^{\alpha x} \sin(\beta x)$	$A e^{\alpha x} \cos(\beta x) + B e^{\alpha x} \sin(\beta x)$
7. $\left(\sum_{j=0}^{n} c_j x^j \right) \cos(\alpha x)$	$\left(\sum_{j=0}^{n} A_j x^j \right) \cos(\alpha x) + \left(\sum_{j=0}^{n} B_j x^j \right) \sin(\alpha x)$
8. $\left(\sum_{j=0}^{n} c_j x^j \right) \sin(\alpha x)$	$\left(\sum_{j=0}^{n} A_j x^j \right) \cos(\alpha x) + \left(\sum_{j=0}^{n} B_j x^j \right) \sin(\alpha x)$
9. $\left(\sum_{j=0}^{n} c_j x^j \right) e^{\alpha x}$	$\left(\sum_{j=0}^{n} A_j x^j \right) e^{\alpha x}$
10. $\left(\sum_{j=0}^{n} c_j x^j \right) e^{\alpha x} \cos(\beta x)$	$\left(\sum_{j=0}^{n} A_j x^j \right) e^{\alpha x} \cos(\beta x) + \left(\sum_{j=0}^{n} B_j x^j \right) e^{\alpha x} \sin(\beta x)$
11. $\left(\sum_{j=0}^{n} c_j x^j \right) e^{\alpha x} \sin(\beta x)$	$\left(\sum_{j=0}^{n} A_j x^j \right) e^{\alpha x} \cos(\beta x) + \left(\sum_{j=0}^{n} B_j x^j \right) e^{\alpha x} \sin(\beta x)$

3) Plug the assumed form of y_p into the original nonhomogeneous equation and determine the coefficients.

4) Assemble the general solution: $y = y_h + y_p$.

Step 4: (Variation of Parameters) ***Is this a general linear nonhomogeneous equation?***

Express the equation in the form below.

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + P(x)\frac{\mathrm{d}y}{\mathrm{d}x} + Q(x)y = g(x)$$

1) Determine the homogeneous solution $y_h = c_1y_1 + c_2y_2$ (i.e. ignore $g(x)$); use the method of ***step 2*** if the equation has constant coefficients or the method of ***step 5*** if it is a Cauchy-Euler equation.

2) Assuming the particular solution has the form $y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$, compute the Wronskians W, W_1, and W_2 below.

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}, \quad W_1 = \begin{vmatrix} 0 & y_2 \\ g(x) & y_2' \end{vmatrix}, \quad \text{and} \quad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & g(x) \end{vmatrix}.$$

3) Solve for u_1 and u_2 from

$$u_1' = \frac{W_1}{W} \quad \text{and} \quad u_2' = \frac{W_2}{W}.$$

4) Assemble the general solution: $y = c_1y_1 + c_2y_2 + u_1y_1 + u_2y_2$.

* Know how this method generalizes to higher-order linear nonhomogeneous ODEs.

Step 5: ***Is this a Cauchy-Euler equation?***

Express the equation in the form below.

$$a_2x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + a_1x\frac{\mathrm{d}y}{\mathrm{d}x} + a_0y = 0$$

1) Using $y = x^m$, establish the characteristic polynomial.

$$a_2m^2 + (a_1 - a_2)m + a_0 = 0$$

2) Determine the roots of the characteristic polynomial m_1 and m_2.

3) Establish the homogeneous solution based on the following three cases:

a) Distinct Roots (m_1 and m_2): $y_h = c_1x^{m_1} + c_2x^{m_2}$

b) Double Root ($m = m_1 = m_2$): $y_h = c_1x^m + c_2x^m\ln(x)$

c) Complex Conjugate Roots ($m_{1,2} = \alpha \pm \beta i$): $y_h = x^\alpha \left[c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x))\right]$

* Know how this method generalizes to higher-order Cauchy-Euler equations.

REFERENCES

Zill, D. G., & Wright, W. S. (2014). *Advanced Engineering Mathematics* (5th ed.). Burlington, MA: Jones & Bartlett Learning.