Lecture V

Exact Equations

In this lecture, we will learn to solve another first-order ODE. In general, when we have some function of two variables defined by $f(x, y) = 0$, we can define its differential as

$$\frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y = 0. \tag{1}$$

This is by definition a first-order ODE, and we could therefore ask the inverse problem, namely given an equation of the form

$$M(x, y)\mathrm{d}x + N(x, y)\mathrm{d}y = 0, \tag{2}$$

can we find some function $f(x, y) = 0$ (an implicit solution of the ODE) such that $M(x, y)$ and $N(x, y)$ correspond to its differential, i.e.

$$M(x, y) = \frac{\partial f}{\partial x} \quad \text{and} \quad N(x, y) = \frac{\partial f}{\partial y}. \tag{3}$$

Such a differential equation, where the functions $M(x, y)$ and $N(x, y)$ correspond to the differential of some function $f(x, y)$, is called an **exact equation**.

How can we recognize that an equation is exact though? Let's look a little deeper. If an equation were exact, we know that

$$\begin{aligned} M(x, y)\mathrm{d}x + N(x, y)\mathrm{d}y &= \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y \\ \therefore \quad M(x, y) = \frac{\partial f}{\partial x} \quad &\text{and} \quad N(x, y) = \frac{\partial f}{\partial y}. \end{aligned}$$

If we take the partial derivative of $M(x, y)$ with respect to y and the partial derivative of $N(x, y)$ with respect to x, we see that

$$\frac{\partial M}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\partial^2 f}{\partial x \partial y},$$

in other words, a necessary and sufficient condition for our differential equation to be exact is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \tag{4}$$

We can now look at how we would solve such an ODE. So given the ODE

$$M(x,y)\mathrm{d}x + N(x,y)\mathrm{d}y = 0,$$

we must first check if the equation is exact by checking if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

If this is true, we can start solving. Let's begin with the relation below (we could have also started with the relation for $N(x,y)$)

$$M(x,y) = \frac{\partial f}{\partial x}.$$

We can integrate both sides with respect to x, and explicitly show the constant of integration (in this case the "constant" could be a function of y, and so we cannot ignore that)

$$f(x,y) = \int M(x,y)\mathrm{d}x + g(y).$$

We can now use this f in the relation

$$N(x,y) = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\int M(x,y)\mathrm{d}x + g'(y).$$

From here, we can solve for the constant of integration giving

$$g(y) = \int \left(N(x,y) - \frac{\partial}{\partial y}\int M(x,y)\mathrm{d}x \right) \mathrm{d}y + c.$$

Finally, we can substitute this in our expression for $f(x, y)$,

$$f(x,y) = \int M(x,y)\mathrm{d}x + \int \left(N(x,y) - \frac{\partial}{\partial y} \int M(x,y)\mathrm{d}x \right) \mathrm{d}y + c. \tag{5}$$

Equivalently, if we would have started with $N(x, y)$, we would have obtained

$$f(x,y) = \int N(x,y)\mathrm{d}y + \int \left(M(x,y) - \frac{\partial}{\partial x} \int N(x,y)\mathrm{d}y \right) \mathrm{d}x + c. \tag{6}$$

A third method of solving the ODE (2) when it is exact is to simply integrate the two expressions in (3), namely

$$f(x,y) = \int M(x,y)\mathrm{d}x + g(y) + c \quad \text{and} \quad f(x,y) = \int N(x,y)\mathrm{d}y + h(x) + c,$$

and then compare both results to determine which terms correspond to $g(y)$ and which correspond to $h(x)$.

EXAMPLE

$$\text{Solve} \quad 3x^2y\mathrm{d}x + \left(x^3 - y^2\right)\mathrm{d}y = 0.$$

First we check if the equation is exact

$$\frac{\partial M}{\partial y} = 3x^2 \quad \text{and} \quad \frac{\partial N}{\partial x} = 3x^2 \quad \therefore \text{ It is exact!}$$

Now we proceed with the solution

$$\frac{\partial f}{\partial x} = M(x,y) \to f(x,y) = \int 3x^2y\mathrm{d}x = x^3y + g(y).$$

We can now use the other relation to find $g(y)$,

$$N(x,y) = \frac{\partial f}{\partial y} \to \left(x^3 - y^2\right) = x^3 + g'(y) \to g(y) = -\frac{1}{3}y^3 + c.$$

The solution is then

$$f(x,y) = x^3y - \frac{1}{3}y^3 + c = 0 \quad \text{or} \quad x^3y - \frac{1}{3}y^3 = c$$

Sometimes, an equation that is not exact can be made exact using an integrating factor. For instance, given the prototype equation

$$M(x,y)\mathrm{d}x + N(x,y)\mathrm{d}y = 0 \quad \text{with} \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x},$$

we can multiply by a function $\mu(x,y)$,

$$\mu(x,y)M(x,y)\mathrm{d}x + \mu(x,y)N(x,y)\mathrm{d}y = 0. \tag{7}$$

In order for this, then, to be an exact equation, we require that

$$\begin{aligned} \frac{\partial}{\partial y}\left[\mu(x,y)M(x,y)\right] &= \frac{\partial}{\partial x}\left[\mu(x,y)N(x,y)\right] \\ \mu\frac{\partial M}{\partial y} + M\frac{\partial \mu}{\partial y} &= \mu\frac{\partial N}{\partial x} + N\frac{\partial \mu}{\partial x} \\ N\frac{\partial \mu}{\partial x} - M\frac{\partial \mu}{\partial y} + \mu\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) &= 0. \end{aligned}$$

This last equation is a partial differential equation, and its solution is outside the scope of this course. Within the scope of this course however, we can distinguish two cases, namely when $\mu = \mu(x)$ and when $\mu = \mu(y)$. In the first case, using the subscript notation for partial derivatives,

$$\mu = \mu(x)$$

$$N\frac{\mathrm{d}\mu}{\mathrm{d}x} + \mu\left(N_x - M_y\right) = 0$$

This is a linear first-order equation in μ!

$$\mu = e^{\int \frac{M_y - N_x}{N}\mathrm{d}x} \quad \text{iff} \quad \frac{M_y - N_x}{N} \neq f(y).$$

In the second case,

$$\mu = \mu(y)$$
$$-M\frac{\mathrm{d}\mu}{\mathrm{d}y} + \mu\left(N_x - M_y\right) = 0$$

This is also a linear first-order equation in μ!

$$\mu = e^{\int \frac{N_x - M_y}{M}\mathrm{d}y} \quad \text{iff} \quad \frac{N_x - M_y}{M} \neq f(x).$$

It is important to note the conditions in each case though, since if the assumption that $\mu = \mu(x)$ or $\mu = \mu(y)$ is violated, the method fails.

Lecture Problems (§2.4): 26, 37

Tutorial Problems (§2.4): 7, 18, 25, 38

Suggested Problems (§2.4): 9, 17, 21, 29, 31

BONUS NOTES

The method of solving exact equations ultimately boils down to a procedure.

Can the equation be expressed in the form below?

$$M(x,y)\mathrm{d}x + N(x,y)\mathrm{d}y = 0 \quad \text{where} \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If not, can I make the equation exact after multiplying through by an integrating factor (μ)?

$$\mu(x) = e^{\int \frac{M_y - N_x}{N}\mathrm{d}x} \quad \text{iff} \quad \frac{M_y - N_x}{N} \neq f(y) \quad \text{or} \quad \mu(y) = e^{\int \frac{N_x - M_y}{M}\mathrm{d}y} \quad \text{iff} \quad \frac{N_x - M_y}{M} \neq f(x).$$

1) Compute the solution $f(x,y)$ by integrating $M(x,y)$ with respect to x.

$$\frac{\partial f}{\partial x} = M(x,y) \to f(x,y) = \int M(x,y)\mathrm{d}x + g(y)$$

2) Compute the partial derivative of the $f(x,y)$ obtained in step (1) with respect to y and determine $g(y)$ using $N(x,y)$.

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\int M(x,y)\mathrm{d}x + \frac{\mathrm{d}g}{\mathrm{d}y} = N(x,y) \to \text{Solve for } g(y).$$

3) Assemble the solution (replace the $g(y)$ in step (1) with what was found in step (2)).

$$f(x,y) = \int M(x,y)\mathrm{d}x + g(y) + c = 0 \quad \text{or} \quad \int M(x,y)\mathrm{d}x + g(y) = c$$

4) Solve for y (if possible) and simplify the result.

REFERENCES

Zill, D. G., & Wright, W. S. (2014). *Advanced Engineering Mathematics* (5th ed.). Burlington, MA: Jones & Bartlett Learning.