Lecture XX

Homogeneous Linear Systems

In this lecture, we will look at how to solve the homogeneous linear system $\mathbf{X}' = \mathbf{AX}$. We will take a similar approach as was done for homogeneous linear nth-order ODEs with constant coefficients, namely assume the solution $\mathbf{X} = \mathbf{K}e^{\lambda t}$, which gives

$$\mathbf{X}' = \mathbf{AX}$$
$$\lambda\mathbf{K}e^{\lambda t} = \mathbf{AK}e^{\lambda t}$$
$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{K} = 0.$$

This is a classic eigenvalue problem. Nontrivial solutions $\mathbf{K} \neq \mathbf{0}$ to this problem will exist only for values of λ such that $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$, called the **characteristic equation**. The solutions, λ, to the characteristic equation are called the eigenvalues, and the corresponding vectors $\mathbf{K}$ are called the eigenvectors. We can distinguish three distinct cases for the solution depending on the eigenvalues.

In the first case, when the $n \times n$ matrix $\mathbf{A}$ possesses n distinct real eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$, then a set of n linearly independent eigenvectors $\mathbf{K}_1, \mathbf{K}_2, ..., \mathbf{K}_n$ can always be found and $\mathbf{X}_1 = \mathbf{K}_1 e^{\lambda_1 t}, \mathbf{X}_2 = \mathbf{K}_2 e^{\lambda_2 t}, ..., \mathbf{X}_n = \mathbf{K}_n e^{\lambda_n t}$ is a fundamental set of solutions of $\mathbf{X}' = \mathbf{AX}$. The general solution would then be

$$\mathbf{X} = c_1\mathbf{K}_1 e^{\lambda_1 t} + c_2\mathbf{K}_2 e^{\lambda_2 t} + \cdots + c_n\mathbf{K}_n e^{\lambda_n t}. \tag{1}$$

EXAMPLE

Solve the following system of ODEs

$$\mathbf{X}' = \begin{bmatrix} 1 & -4 \\ 1 & 6 \end{bmatrix}\mathbf{X}.$$

In solving such a system, we must first determine the eigenvalues of the matrix $\mathbf{A}$, namely

$$\begin{vmatrix} 1-\lambda & -4 \\ 1 & 6-\lambda \end{vmatrix} = \lambda^2 - 7\lambda + 10 = (\lambda-2)(\lambda-5) = 0.$$

This system therefore has the two distinct eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 5$. We can then find their corresponding eigenvectors from $(\mathbf{A} - \lambda_j \mathbf{I})\mathbf{K}_j = 0$, giving

$$\mathbf{K}_1 = \begin{bmatrix} 4 \\ -1 \end{bmatrix} \quad \text{and} \quad \mathbf{K}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

The general solution is then

$$\mathbf{X} = c_1 \begin{bmatrix} 4 \\ -1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{5t}.$$

In the second case, when we have a repeated eigenvalue, we have to consider two sub-cases. 1) For some $n \times n$ matrices $\mathbf{A}$, it may be possible to find m linearly independent eigenvectors $\mathbf{K}_1, \mathbf{K}_2, ..., \mathbf{K}_m$ corresponding to a single eigenvalue λ_1 of multiplicity $m \leq n$. In this case, the general solution of the system contains the linear combination

$$c_1 \mathbf{K}_1 e^{\lambda_1 t} + c_2 \mathbf{K}_2 e^{\lambda_1 t} + \cdots + c_m \mathbf{K}_m e^{\lambda_1 t}. \tag{2}$$

2) If there is only one eigenvector corresponding to the eigenvalue λ_1 of multiplicity m, then m linearly independent solutions of the form

$$\begin{aligned} \mathbf{X}_1 &= \mathbf{K}_1 e^{\lambda_1 t} \\ \mathbf{X}_2 &= \mathbf{K}_1 t e^{\lambda_1 t} + \mathbf{K}_2 e^{\lambda_1 t} \\ &\cdots \\ \mathbf{X}_m &= \mathbf{K}_1 \tfrac{t^{m-1}}{(m-1)!} e^{\lambda_1 t} + \mathbf{K}_2 \tfrac{t^{m-2}}{(m-2)!} e^{\lambda_1 t} + \cdots + \mathbf{K}_m e^{\lambda_1 t}, \end{aligned} \tag{3}$$

where $\mathbf{K}_j$ are column vectors, can always be found. We can obtain each $\mathbf{K}_j$ by solving each of the following one by one

$$
\begin{aligned}
&(\mathbf{A}-\lambda\mathbf{I})\mathbf{K}_1 = 0 \\
&(\mathbf{A}-\lambda\mathbf{I})\mathbf{K}_2 = \mathbf{K}_1 \\
&(\mathbf{A}-\lambda\mathbf{I})\mathbf{K}_3 = \mathbf{K}_2 \\
&\ldots \\
&(\mathbf{A}-\lambda\mathbf{I})\mathbf{K}_m = \mathbf{K}_{m-1}.
\end{aligned}
$$

EXAMPLE

Solve the following system of ODEs

$$\mathbf{X}' = \begin{bmatrix} -1 & -2 \\ 2 & 3 \end{bmatrix} \mathbf{X}.$$

The eigenvalues of the above system are obtained from

$$\begin{vmatrix} -1-\lambda & -2 \\ 2 & 3-\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 1 = 0,$$

giving $\lambda_{1,2} = 1$. There is only one eigenvector (which of course can be multiplied by any constant) associated to $\lambda = 1$, namely

$$\mathbf{K}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Since there are no other distinct eigenvectors that can be found for $\lambda = 1$, we must search for the vector $\mathbf{K}_2$ from $(\mathbf{A}-\lambda\mathbf{I})\mathbf{K}_2 = \mathbf{K}_1$, i.e.

$$\begin{bmatrix} -2 & -2 \\ 2 & 2 \end{bmatrix} \mathbf{K}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \to \mathbf{K}_2 = \begin{bmatrix} 0 \\ -\frac{1}{2} \end{bmatrix}.$$

The general solution is then given by

$$\mathbf{X} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t + c_2 \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} t + \begin{bmatrix} 0 \\ -\frac{1}{2} \end{bmatrix} \right) e^t.$$

The third case is when we have complex conjugate eigenvalues $\lambda_{1,2} = \alpha \pm i\beta$. Following directly from the first case, we can deduce the following theorem.

THEOREM: Solutions Corresponding to a Complex Eigenvalue

"Let $\mathbf{A}$ be the coefficient matrix having real entries of the homogeneous system $\mathbf{X}' = \mathbf{A}\mathbf{X}$, and let $\mathbf{K}_1$ be an eigenvector corresponding to the complex eigenvalue $\lambda_1 = \alpha + i\beta$, α and β real. Then

$$\mathbf{K}_1 e^{\lambda_1 t} \quad \text{and} \quad \bar{\mathbf{K}}_1 e^{\bar{\lambda}_1 t}$$

are solutions of $\mathbf{X}' = \mathbf{A}\mathbf{X}$." (Zill & Wright, 2014)

We therefore have the two solutions

$$\mathbf{K}_1 e^{\lambda_1 t} = \mathbf{K}_1 e^{\alpha t}(\cos(\beta t) + i \sin(\beta t))$$
$$\bar{\mathbf{K}}_1 e^{\bar{\lambda}_1 t} = \bar{\mathbf{K}}_1 e^{\alpha t}(\cos(\beta t) - i \sin(\beta t)).$$

Owing to the superposition principle, we can rewrite these solutions as

$$\mathbf{X}_1 = \frac{1}{2}(\mathbf{K}_1 e^{\lambda_1 t} + \bar{\mathbf{K}}_1 e^{\bar{\lambda}_1 t}) = \frac{1}{2}(\mathbf{K}_1 + \bar{\mathbf{K}}_1) e^{\alpha t} \cos(\beta t) - \frac{i}{2}(-\mathbf{K}_1 + \bar{\mathbf{K}}_1) e^{\alpha t} \sin(\beta t)$$
$$\mathbf{X}_2 = \frac{i}{2}(-\mathbf{K}_1 e^{\lambda_1 t} + \bar{\mathbf{K}}_1 e^{\bar{\lambda}_1 t}) = \frac{i}{2}(-\mathbf{K}_1 + \bar{\mathbf{K}}_1) e^{\alpha t} \cos(\beta t) + \frac{1}{2}(\mathbf{K}_1 + \bar{\mathbf{K}}_1) e^{\alpha t} \sin(\beta t).$$

Noting that $\text{Re}(\mathbf{K}_1) = \frac{1}{2}(\mathbf{K}_1 + \bar{\mathbf{K}}_1)$ and that $\text{Im}(\mathbf{K}_1) = \frac{i}{2}(-\mathbf{K}_1 + \bar{\mathbf{K}}_1)$, the two corresponding real solutions can therefore be expressed as

$$\mathbf{X_1} = e^{\alpha t}[\mathbf{B}_1 \cos(\beta t) - \mathbf{B}_2 \sin(\beta t)]$$
$$\mathbf{X_2} = e^{\alpha t}[\mathbf{B}_2 \cos(\beta t) + \mathbf{B}_1 \sin(\beta t)] \tag{4}$$
$$\text{where} \quad \mathbf{B}_1 = \text{Re}(\mathbf{K}_1) \quad \text{and} \quad \mathbf{B}_2 = \text{Im}(\mathbf{K}_1).$$

It is simpler however not to remember the above formula, but rather to treat the problem as was done in the first case and then to make use of Euler's identity (i.e. $e^{\pm i\beta t} = \cos(\beta t) \pm i\sin(\beta t)$).

EXAMPLE

Solve the following system of ODEs

$$\mathbf{X}' = \begin{bmatrix} 1 & 4 \\ -1 & 1 \end{bmatrix} \mathbf{X}.$$

Again, we begin by finding the eigenvalues from

$$\begin{vmatrix} 1-\lambda & 4 \\ -1 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 5 = 0,$$

giving $\lambda_{1,2} = 1 \pm 2i$. From here, we can determine the eigenvectors

$$\mathbf{K}_1 = \begin{bmatrix} 2 \\ i \end{bmatrix} \quad \text{and} \quad \mathbf{K}_2 = \begin{bmatrix} 2 \\ -i \end{bmatrix}.$$

The general solution would then be

$$\mathbf{X} = e^t \left\{ c_1 \begin{bmatrix} 2 \\ i \end{bmatrix} e^{2it} + c_2 \begin{bmatrix} 2 \\ -i \end{bmatrix} e^{-2it} \right\},$$

however since we will only be interested in the real part, let us use Euler's identity

$$\mathbf{X} = e^t \left\{ \tilde{c}_1 \begin{bmatrix} 2 \\ i \end{bmatrix} [\cos(2t) + i\sin(2t)] + \tilde{c}_2 \begin{bmatrix} 2 \\ -i \end{bmatrix} [\cos(2t) - i\sin(2t)] \right\}.$$

Upon multiplying out we obtain

$$\mathbf{X} = e^t \left\{ \tilde{c}_1 \left(\begin{bmatrix} 2\cos(2t) \\ -\sin(2t) \end{bmatrix} + i \begin{bmatrix} 2\sin(2t) \\ \cos(2t) \end{bmatrix} \right) + \tilde{c}_2 \left(\begin{bmatrix} 2\cos(2t) \\ -\sin(2t) \end{bmatrix} - i \begin{bmatrix} 2\sin(2t) \\ \cos(2t) \end{bmatrix} \right) \right\},$$

or

$$\mathbf{X} = e^t \left\{ (\tilde{c}_1 + \tilde{c}_2) \begin{bmatrix} 2\cos(2t) \\ -\sin(2t) \end{bmatrix} + i(\tilde{c}_1 - \tilde{c}_2) \begin{bmatrix} 2\sin(2t) \\ \cos(2t) \end{bmatrix} \right\}.$$

Since we are interested only in the real part, relabeling $c_1 = \mathrm{Re}(\tilde{c}_1 + \tilde{c}_2)$ and $c_2 = \mathrm{Re}(i(\tilde{c}_1 - \tilde{c}_2))$, we have as the general solution

$$\mathbf{X} = e^t \left\{ c_1 \begin{bmatrix} 2\cos(2t) \\ -\sin(2t) \end{bmatrix} + c_2 \begin{bmatrix} 2\sin(2t) \\ \cos(2t) \end{bmatrix} \right\}.$$

Note that the answer is the same as if we applied the form of equation (4) directly, i.e.

$$\mathbf{X} = c_1 e^t \left(\begin{bmatrix} 2 \\ 0 \end{bmatrix} \cos(2t) - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin(2t) \right) + c_2 e^t \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos(2t) + \begin{bmatrix} 2 \\ 0 \end{bmatrix} \sin(2t) \right).$$

Lecture Problems (§10.2): 2, 24, 36

Tutorial Problems (§10.2): 3, 14, 22, 25, 38

Suggested Problems (§10.2): 1, 13, 21, 23, 35, 37

BONUS NOTES

None.

REFERENCES

Zill, D. G., & Wright, W. S. (2014). *Advanced Engineering Mathematics* (5th ed.). Burlington, MA: Jones & Bartlett Learning.