Lecture III

Separable Equations

In this lecture, we will learn our first method to solve ordinary differential equations. In the case of a first-order ODE, not necessarily linear in the dependent variable (y), of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y) \tag{1}$$

we can distinguish a particular case where the function $f(x, y)$ can be expressed as the product of two separate functions of x and y alone, namely where we can write $f(x, y) = g(x)h(y)$. This is known as a **separable first-order ODE**, and we may proceed with its solution as follows:

Given a separable first-order ODE

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y),$$

we will be able to write $f(x, y) = g(x)h(y)$, giving

$$\frac{\mathrm{d}y}{\mathrm{d}x} = g(x)h(y). \tag{2}$$

We can divide by $h(y)$ on both sides

$$\frac{1}{h(y)}\frac{\mathrm{d}y}{\mathrm{d}x} = g(x) \quad \text{or more simply} \quad p(y)\frac{\mathrm{d}y}{\mathrm{d}x} = g(x)$$

and then integrate both sides with respect to x

$$\int p(y)\frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x = \int g(x)\mathrm{d}x.$$

Noting that y is in fact a function of x, we can simplify the left-hand side of the above equation since

$$\mathrm{d}y = \frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x,$$

giving

$$\int p(y)\frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x = \int p(y)\mathrm{d}y = \int g(x)\mathrm{d}x.$$

The equation

$$\int p(y)\mathrm{d}y = \int g(x)\mathrm{d}x + c \tag{3}$$

gives the solution of the separable first-order ODE upon integration (**it is important not to forget the resulting constant of integration**); note that the constant of integration c is explicitly shown here. If the resulting equation can be rearranged to give an expression of the form $y = \phi(x)$, this in an **explicit** solution, if it is not possible to rearrange it in this way, we have an **implicit** solution $\phi(x, y) = 0$.

Ultimately, solving separable first-order ODEs simplifies to rearranging the original ODE such that all the y's and $\mathrm{d}y$'s are on one side of the equation and all the x's and $\mathrm{d}x$'s are on the other (treating $\mathrm{d}y/\mathrm{d}x$ as though it were an algebraic fraction thanks to the chain rule for differentiation).

EXAMPLE

$$\text{Solve} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2}{1+2x}.$$

First, we will separate the x's and y's

$$\frac{\mathrm{d}y}{y^2} = \frac{\mathrm{d}x}{1+2x}$$

and then integrate both sides

$$\int \frac{\mathrm{d}y}{y^2} = \int \frac{\mathrm{d}x}{1+2x}$$

giving

$$-\frac{1}{y} = \frac{1}{2}\ln|1+2x| + \tilde{c}.$$

In this case, we have an explicit solution since we can express y as an explicit function of x

(where $c = 2\tilde{c}$)

$$y = -\frac{2}{\ln|1+2x| + c}.$$

EXAMPLE

$$\text{Solve} \quad (2y-1)\, e^{y^2} \cos(x) \frac{\mathrm{d}y}{\mathrm{d}x} = e^{y} \sin(x) \quad \text{with} \quad y(0) = 0.$$

Again, we start by separating the x's and y's

$$(2y-1)\, e^{y^2-y} \mathrm{d}y = \tan(x) \mathrm{d}x.$$

Integrating both sides then gives

$$e^{y^2-y} = -\ln|\cos(x)| + c.$$

We can then make use of the initial condition $y(0) = 0$ giving $c = 1$. The resulting implicit solution is

$$e^{y^2-y} = -\ln|\cos(x)| + 1.$$

Lecture Problems (§2.2): 19, 28

Tutorial Problems (§2.2): 2, 4, 13, 24, 30

Suggested Problems (§2.2): 7, 9, 23, 25

BONUS NOTES

The method of solving separable equations ultimately boils down to a procedure.

Can the equation be expressed in the form below?

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y) \quad \text{where} \quad f(x, y) = g(x)h(y)$$

1) Put all the y's and $\mathrm{d}y$'s on the left-hand side of the equation.

2) Put all the x's and $\mathrm{d}x$'s on the right-hand side of the equation.

3) Integrate both sides (don't forget the constant of integration on the right-hand side).

4) Solve for y (if possible) and simplify the result.

For the more curious reader, we might ask whether we can generalize the method of separation of variables to the nth-order equation

$$\frac{\mathrm{d}^{(n)}y}{\mathrm{d}x^{(n)}} = f(x, y)$$

in the case where we can express $f(x, y)$ as the product of two separate functions of x and y alone as was learned in this lecture ($f(x, y) = g(x)h(y)$). In fact, this is not at all obvious because we can't treat $\mathrm{d}^{(n)}y/\mathrm{d}x^{(n)}$ as an algebraic fraction as we did for $\mathrm{d}y/\mathrm{d}x$; this was only possible as a consequence of the chain rule in deriving equation (3). There are, however, cases that we can solve by separation of variables after using a substitution. For example, let's look at the second-order ODE

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = f(y).$$

We can use the substitution

$$u = \frac{\mathrm{d}y}{\mathrm{d}x}$$

to reduce the second-order derivative as follows

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}x} = u\frac{\mathrm{d}u}{\mathrm{d}y}.$$

The differential equation then reduces to the separable equation

$$u \frac{\mathrm{d}u}{\mathrm{d}y} = f(y)$$

having solution

$$u = \sqrt{2 \int f(y) \mathrm{d}y + c_1}$$

after which we must solve for y from

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{2 \int f(y) \mathrm{d}y + c_1}.$$

The complete solution (likely implicit in y) would be given by

$$x = \int \frac{\mathrm{d}y}{\sqrt{2 \int f(y) \mathrm{d}y + c_1}} + c_2,$$

where the constants of integration c_1 and c_2 are explicitly shown.

REFERENCES

Zill, D. G., & Wright, W. S. (2014). *Advanced Engineering Mathematics* (5th ed.). Burlington, MA: Jones & Bartlett Learning.