Lecture VI

Solutions by Substitution

For first-order ordinary differential equations, we have learned to look carefully at the equation and decide whether it is separable, linear or exact or whether we can apply some trick to make it so (manipulating the equation, changing the dependent variable and using integrating factors). Here we will learn our last method in solving first-order ODEs, namely using substitutions.

There are three distinct cases we will discuss where we might apply a substitution, however there may be instances where other substitutions will work as well.

First-Order Equations with Homogeneous Functions

In the first case, when a differential equation is comprised of homogeneous functions, we can take advantage of their properties to solve the equation. A function that possesses the property $f(tx, ty) = t^\alpha f(x, y)$ is called a homogeneous function of degree α. Say we have a first-order differential equation of the form

$$M(x, y)\mathrm{d}x + N(x, y)\mathrm{d}y = 0 \tag{1}$$

where both $M(x, y)$ and $N(x, y)$ are homogeneous functions of the **same** degree. If this is the case, we could use the property in reverse

$$\begin{aligned} M(x, y)\mathrm{d}x + N(x, y)\mathrm{d}y &= 0 \\ t^\alpha M(x, y)\mathrm{d}x + t^\alpha N(x, y)\mathrm{d}y &= 0 \\ M(tx, ty)\mathrm{d}x + N(tx, ty)\mathrm{d}y &= 0. \end{aligned}$$

Since t is completely arbitrary, if we use $t = 1/x$ or $t = 1/y$, we reduce the equation to

$$\begin{aligned} t = 1/x &\rightarrow M(1, y/x)\mathrm{d}x + N(1, y/x)\mathrm{d}y = 0 \\ t = 1/y &\rightarrow M(x/y, 1)\mathrm{d}x + N(x/y, 1)\mathrm{d}y = 0. \end{aligned}$$

In each case, it would now make sense to use a substitution. In the first case, we would use $u = y/x$ and in the second $v = x/y$,

$$M(1, y/x)\mathrm{d}x + N(1, y/x)\mathrm{d}y = 0 \to y = ux \quad \text{and} \quad \mathrm{d}y = u\mathrm{d}x + x\mathrm{d}u$$
$$M(x/y, 1)\mathrm{d}x + N(x/y, 1)\mathrm{d}y = 0 \to x = vy \quad \text{and} \quad \mathrm{d}x = v\mathrm{d}y + y\mathrm{d}v.$$

Either substitution will result in a separable equation, namely

$$\frac{\mathrm{d}x}{x} + \frac{N(1,u)\mathrm{d}u}{M(1,u) + uN(1,u)} = 0 \quad \text{or} \quad \frac{\mathrm{d}y}{y} + \frac{M(v,1)\mathrm{d}v}{N(v,1) + vM(v,1)} = 0 \tag{2}$$

EXAMPLE

$$\text{Solve} \quad \left(x^2 + y^2\right)\mathrm{d}x + \left(x^2 - xy\right)\mathrm{d}y = 0$$

We must first check if the functions M and N are homogeneous functions of the same degree by substituting $(x, y) \to (tx, ty)$,

$$x^2 + y^2 \to (tx)^2 + (ty)^2 = t^2\left(x^2 + y^2\right)$$
$$x^2 - xy \to (tx)^2 - (tx)(ty) = t^2\left(x^2 - xy\right),$$

so it turns out that in fact they are homogeneous functions of degree 2. We can therefore use the substitution

$$\left(x^2 + y^2\right)\mathrm{d}x + \left(x^2 - xy\right)\mathrm{d}y = 0$$
$$y = ux \quad \text{and} \quad \mathrm{d}y = u\mathrm{d}x + x\mathrm{d}u$$
$$\left[x^2 + (ux)^2\right]\mathrm{d}x + \left[x^2 - x(ux)\right](u\mathrm{d}x + x\mathrm{d}u) = 0$$
$$x^2(1+u)\mathrm{d}x + x^3(1-u)\mathrm{d}u = 0$$
$$\frac{\mathrm{d}x}{x} = \frac{u-1}{u+1}\mathrm{d}u.$$

We can now integrate to solve the differential equation and substitute back $u = y/x$

$$ln|x| = u - 2ln|u+1| + \tilde{c}$$
$$x = \frac{ce^u}{(u+1)^2}$$
$$(x+y)^2 = cxe^{y/x}$$

Bernoulli's Equation

In the second case, we have the nonlinear first-order ODE known as a Bernoulli equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)y^n, \tag{3}$$

which can be manipulated to lead us to an obvious substitution. For instance, if we multiply the equation by y^{-n}

$$y^{-n}\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y^{1-n} = Q(x).$$

Making use of the chain rule, we see that

$$y^{-n}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{1-n}\frac{\mathrm{d}}{\mathrm{d}x}\left(y^{1-n}\right),$$

therefore

$$\frac{1}{1-n}\frac{\mathrm{d}}{\mathrm{d}x}\left(y^{1-n}\right) + P(x)y^{1-n} = Q(x).$$

It is now more clear that we should use the substitution $u = y^{1-n}$ since it will reduce the equation to a first-order linear equation, i.e.

$$\frac{\mathrm{d}u}{\mathrm{d}x} + (1-n)P(x)u = (1-n)Q(x). \tag{4}$$

EXAMPLE

$$\text{Solve} \quad 2x^2\frac{\mathrm{d}y}{\mathrm{d}x} + y = y^3.$$

This is a Bernoulli equation with $n = 3$. Let's begin, therefore, by multiplying through by

y^{-3}, giving

$$2x^2y^{-3}\frac{\mathrm{d}y}{\mathrm{d}x} + y^{-2} = 1.$$

We can now use the chain rule to simplify the derivative term

$$-x^2\frac{\mathrm{d}}{\mathrm{d}x}\left(y^{-2}\right) + y^{-2} = 1.$$

Using the substitution $u = y^{-2}$ and putting the equation in standard form, we obtain the first-order linear equation

$$\frac{\mathrm{d}u}{\mathrm{d}x} - x^{-2}u = -x^{-2}$$

whose solution is obtained using an integrating factor as follows

$$\begin{gathered}
I(x) = e^{1/x} \\
\frac{\mathrm{d}u}{\mathrm{d}x} - x^{-2}u = -x^{-2} \to e^{1/x}\frac{\mathrm{d}u}{\mathrm{d}x} - x^{-2}e^{1/x}u = -x^{-2}e^{1/x} \\
\frac{\mathrm{d}}{\mathrm{d}x}\left[e^{1/x}u\right] = -x^{-2}e^{1/x} \\
u = 1 + ce^{-1/x} \to y^{-2} = 1 + ce^{-1/x} \\
y = \frac{1}{\left(1 + ce^{-1/x}\right)^2}.
\end{gathered}$$

Equations Reducible to Separable Equations

First-order equations that can be expressed in the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(Ax + By + C) \tag{5}$$

can be reduced to a separable equation using the substitution $u = Ax+By+C$. For instance, the equation would reduce to

$$\frac{1}{B}\frac{\mathrm{d}u}{\mathrm{d}x} - \frac{A}{B} = f(u),$$

or after separation of variables

$$\mathrm{d}x = \frac{\mathrm{d}u}{A + Bf(u)}. \tag{6}$$

Revisiting homogeneous equations, we can use the same logic to discover their solution. Say we have a first-order equation that can be expressed in the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(\frac{y}{x}) \quad \text{or} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = f(\frac{x}{y}), \tag{7}$$

then we can reduce this to a separable equation using the corresponding substitution $u = y/x$ or $v = x/y$. This is in fact another approach to solving homogeneous equations, the derivation is shown below.

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}x} &= \frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{y}{x^2} = \frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{u}{x} \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= x\frac{\mathrm{d}u}{\mathrm{d}x} + u \\ x\frac{\mathrm{d}u}{\mathrm{d}x} + u &= f(u) \\ \frac{\mathrm{d}x}{x} &= \frac{\mathrm{d}u}{f(u) - u} \end{aligned}$$

or in the latter case

$$\begin{aligned} \frac{\mathrm{d}v}{\mathrm{d}x} &= -\frac{x}{y^2}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{y} = -\frac{v}{y}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{y} \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{v}\left(1 - \frac{x}{v}\frac{\mathrm{d}v}{\mathrm{d}x}\right) \\ \frac{1}{v}\left(1 - \frac{x}{v}\frac{\mathrm{d}v}{\mathrm{d}x}\right) &= f(v) \\ \frac{\mathrm{d}x}{x} &= \frac{\mathrm{d}v}{v(1 - vf(v))}. \end{aligned}$$

Lecture Problems (§2.5): 1, 11

Tutorial Problems (§2.5): 6, 7, 12, 20

Suggested Problems (§2.5): 5, 13, 19, 23, 27

BONUS NOTES

The method of solving first-order equations composed of homogeneous functions ultimately boils down to a procedure.

Can the equation be expressed in the form below?

$$M(x,y)\mathrm{d}x + N(x,y)\mathrm{d}y = 0$$

where $M(x,y)$ and $N(x,y)$ are homogeneous functions of the same degree; recall that a homogeneous function possesses the property $f(tx,ty) = t^{\alpha} f(x,y)$.

1) Use one of the substitutions below.

$$\begin{aligned} &\text{a)} \quad y = ux \quad \text{and} \quad \mathrm{d}y = u\mathrm{d}x + x\mathrm{d}u \\ &\text{b)} \quad x = vy \quad \text{and} \quad \mathrm{d}x = v\mathrm{d}y + y\mathrm{d}v. \end{aligned}$$

2) The equation is now a separable equation, so solve for u or v.

3) Substitute back $u = y/x$ or $v = x/y$.

4) Solve for y (if possible) and simplify the result.

Likewise, the method of solving Bernoulli equations ultimately boils down to a procedure as well. Can the equation be expressed in the form below?

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)y^n$$

If not, does the equation become a Bernoulli equation when we interchange the dependent and independent variables?

1) Determine the value of n.

2) Using the substitution $u = y^{1-n}$, we obtain the linear equation:

$$\frac{\mathrm{d}u}{\mathrm{d}x} + (1-n)P(x)u = (1-n)Q(x)$$

3) Solve for u and substitute back $u = y^{1-n}$.

4) Solve for y and simplify the result.

REFERENCES

Zill, D. G., & Wright, W. S. (2014). *Advanced Engineering Mathematics* (5th ed.). Burlington, MA: Jones & Bartlett Learning.