Lecture XIX

Theory of Linear Systems

Much of what was learned for linear ODEs apply just as well to systems of linear ODEs, with the exception that the language we use shifts from basic algebra to matrix algebra. We will begin by concerning ourselves with first-order linear systems of n ODEs in normal form

$$\begin{aligned}
x_1' &= a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n + f_1(t)\\
x_2' &= a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n + f_2(t)\\
\cdots&\\
x_n' &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n + f_n(t).
\end{aligned}$$

More conveniently, we could express this system in matrix form

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{bmatrix} = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix}$$

or simply as

$$\mathbf{X}' = \mathbf{A}\mathbf{X} + \mathbf{F}. \tag{1}$$

Here, $\mathbf{X}$ is the solution vector whose entries are differentiable functions satisfying the above system on some interval I. In the case of a homogeneous linear system of n ODEs, we have $\mathbf{F} = \mathbf{0}$, namely

$$\mathbf{X}' = \mathbf{A}\mathbf{X}. \tag{2}$$

We can also define an initial value problem as

$$\mathbf{X}' = \mathbf{A}\mathbf{X} + \mathbf{F} \quad \text{subject to} \quad \mathbf{X}(t_0) = \mathbf{X}_0. \tag{3}$$

Please find below several definitions and theorems for linear systems of ODEs.

THEOREM: Existence of a Unique Solution

"Let the entries of the matrices $\mathbf{A}(t)$ and $\mathbf{F}(t)$ be functions continuous on a common interval I that contains the point t_0. Then there exists a unique solution of the initial-value problem (3) on the interval." (Zill & Wright, 2014)

THEOREM: Superposition Principle

"Let $\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_k$ be a set of solution vectors of the homogeneous system (2) on an interval I. Then the linear combination

$$\mathbf{X} = c_1\mathbf{X}_1 + c_2\mathbf{X}_2 + \cdots + c_k\mathbf{X}_k$$

where the c_j, $j = 1, 2, ..., k$ are arbitrary constants, is also a solution on the interval." (Zill & Wright, 2014)

DEFINITION: Linear Dependence & Independence

"Let $\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_k$ be a set of solution vectors of the homogeneous system (2) on an interval I. We say that the set is **linearly dependent** on the interval if there exist constants $c_1, c_2, ..., c_k$, not all zero, such that

$$c_1\mathbf{X}_1 + c_2\mathbf{X}_2 + \cdots + c_k\mathbf{X}_k = \mathbf{0}$$

for every t in the interval. If the set of vectors is not linearly dependent on the interval, it is said to be **linearly independent**." (Zill & Wright, 2014)

THEOREM: Criterion for Linearly Independent Solutions

"Let

$$\mathbf{X}_1 = \begin{bmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{bmatrix}, \quad \mathbf{X}_2 = \begin{bmatrix} x_{12} \\ x_{22} \\ \vdots \\ x_{n2} \end{bmatrix}, \quad \cdots, \mathbf{X}_n = \begin{bmatrix} x_{1n} \\ x_{2n} \\ \vdots \\ x_{nn} \end{bmatrix}$$

be n solution vectors of the homogeneous system (2) on an interval I. Then the set of solution vectors is linearly independent on I if and only if the **Wronskian**

$$W(\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_n) = \begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{vmatrix} \neq 0$$

for every t in the interval." (Zill & Wright, 2014)

DEFINITION: Fundamental Set of Solutions

"Any set $\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_n$ of n linearly independent solution vectors of the homogeneous system (2) on an interval I is said to be a **fundamental set of solutions** on the interval." (Zill & Wright, 2014)

THEOREM: Existence of a Fundamental Set of Solutions

"There exists a fundamental set of solutions for the homogeneous system (2) on an interval I." (Zill & Wright, 2014)

THEOREM: General Solution of Homogeneous Systems

"Let $\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_n$ be a fundamental set of solutions of the homogeneous system (2) on an interval I. Then the **general solution** of the system on the interval is

$$\mathbf{X} = c_1\mathbf{X}_1 + c_2\mathbf{X}_2 + \cdots + c_n\mathbf{X}_n$$

where the c_j, $j = 1, 2, ..., n$ are arbitrary constants." (Zill & Wright, 2014)

THEOREM: General Solution of Nonhomogeneous Systems

"Let $\mathbf{X}_p$ be a given solution of the nonhomogeneous system (1) on an interval I, and let

$$\mathbf{X}_h = c_1\mathbf{X}_1 + c_2\mathbf{X}_2 + \cdots + c_n\mathbf{X}_n$$

denote the general solution on the same interval of the associated homogeneous system (2). Then the **general solution** of the nonhomogeneous system on the interval is

$$\mathbf{X} = \mathbf{X}_h + \mathbf{X}_p.$$

The general solution $\mathbf{X}_h$ of the associated homogeneous system (2) is called the **homogeneous solution** or **complementary function** of the nonhomogeneous system (1)." (Zill & Wright, 2014)

Lecture Problems (§10.1): 3, 14
Tutorial Problems (§10.1): 2, 6, 15, 18
Suggested Problems (§10.1): 1, 5, 11, 13

BONUS NOTES

None.

REFERENCES

Zill, D. G., & Wright, W. S. (2014). *Advanced Engineering Mathematics* (5th ed.). Burlington, MA: Jones & Bartlett Learning.