Lecture XIV

The Cauchy-Euler Equation

Coming back to homogeneous equations for a moment, we learned to search for solutions of the form $y = e^{mx}$ in the case of a homogeneous linear nth-order ODE with constant coefficients. Here we will see how to search for solutions for a particular homogeneous linear nth-order ODE called the Cauchy-Euler equation,

$$a_n x^n \frac{\mathrm{d}^n y}{\mathrm{d}x^n} + a_{n-1} x^{n-1} \frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \cdots + a_1 x \frac{\mathrm{d}y}{\mathrm{d}x} + a_0 y = 0. \tag{1}$$

For simplicity, let us look at the second-order Cauchy-Euler equation

$$ax^2 y'' + bxy' + cy = 0. \tag{2}$$

In this case, we will search for solutions of the form $y = x^m$, giving

$$\begin{aligned} ax^2 y'' + bxy' + cy &= 0 \\ ax^2 (x^m)'' + bx(x^m)' + c(x^m) &= 0 \\ am(m-1)x^m + bmx^m + cx^m &= 0 \\ am^2 + (b-a)m + c &= 0. \end{aligned}$$

As before, we obtain an auxiliary equation in m and the form of the homogeneous solution will depend on whether the roots of the auxiliary equation are distinct, equal, or complex.

Let us consider each case one by one. In the first case, m_1 and m_2 are distinct, therefore both the solutions $y_1 = c_1 x^{m_1}$ and $y_2 = c_2 x^{m_2}$ must form the general solution. In the higher-order case, each distinct root m_j will contribute a solution $y_j = c_j x^{m_j}$.

$$\text{Case 1: Distinct Real Roots} \rightarrow y = c_1 x^{m_1} + c_2 x^{m_2}. \tag{3}$$

In the second case, both the roots are equal, i.e. $m_1 = m_2 = m$. In this case, we know that $y_1 = c_1 x^m$ must be a solution, however we are missing the second solution. It can be obtained via the reduction of order technique, namely

$$y_2 = x^m \int \frac{e^{-(b/a)\ln(x)}}{x^{2m}} \mathrm{d}x = x^m \ln(x).$$

The general solution is therefore

$$\text{Case 2: Repeated Real Roots} \rightarrow y = c_1 x^m + c_2 x^m \ln(x). \tag{4}$$

For higher-order cases, if a root m has multiplicity k, then the corresponding k linearly independent solutions to be included are

$$x^m, \quad x^m \ln(x), \quad x^m (\ln(x))^2, \quad \cdots, \quad x^m (\ln(x))^{k-1}$$

Finally, we arrive at the third case, and as before it requires more work. In this case, the roots are complex conjugates $m_{1,2} = \alpha \pm i\beta$ and so we have the general solution $y = \tilde{c}_1 x^{\alpha+i\beta} + \tilde{c}_2 x^{\alpha-i\beta}$, where $\tilde{c}_1$ and $\tilde{c}_2$ are complex. We will again work to extract only the real part

$$\begin{gathered}
y = \tilde{c}_1 x^{\alpha+i\beta} + \tilde{c}_2 x^{\alpha-i\beta} \\
y = x^\alpha \left(\tilde{c}_1 x^{i\beta} + \tilde{c}_2 x^{-i\beta}\right) \\
y = x^\alpha \left(\tilde{c}_1 e^{i\beta\ln(x)} + \tilde{c}_2 e^{-i\beta\ln(x)}\right) \\
y = x^\alpha \left[\tilde{c}_1 \left(\cos(\beta\ln(x)) + i\sin(\beta\ln(x))\right) + \tilde{c}_2 \left(\cos(\beta\ln(x)) - i\sin(\beta\ln(x))\right)\right] \\
y = x^\alpha \left[(\tilde{c}_1 + \tilde{c}_2)\cos(\beta\ln(x)) + i(\tilde{c}_1 + \tilde{c}_2)\sin(\beta\ln(x))\right] \\
\mathrm{Re}(y) = x^\alpha \left[\mathrm{Re}(\tilde{c}_1 + \tilde{c}_2)\cos(\beta\ln(x)) + \mathrm{Re}(i(\tilde{c}_1 + \tilde{c}_2))\sin(\beta\ln(x))\right].
\end{gathered}$$

We can simply label the real parts of the constants above as c_1 and c_2 respectively, yielding the general real solution

$$\text{Case 3: Complex Conjugate Roots} \rightarrow y = x^\alpha \left[c_1 \cos(\beta\ln(x)) + c_2 \sin(\beta\ln(x))\right]. \tag{5}$$

For higher-order cases, if a complex conjugate root $\alpha_j \pm i\beta_j$ has multiplicity k, then the corresponding $2k$ linearly independent solutions to be included are

$$x^{\alpha}\cos(\beta\ln(x)),\quad x^{\alpha}\ln(x)\cos(\beta\ln(x)),\quad \cdots,\quad x^{\alpha}(\ln(x))^{k-1}\cos(\beta\ln(x))$$

$$x^{\alpha}\sin(\beta\ln(x)),\quad x^{\alpha}\ln(x)\sin(\beta\ln(x)),\quad \cdots,\quad x^{\alpha}(\ln(x))^{k-1}\sin(\beta\ln(x)).$$

Lecture Problems (§3.6): 2, 20

Tutorial Problems (§3.6): 4, 10, 22, 31

Suggested Problems (§3.6): 1, 5, 11, 19, 21

BONUS NOTES

None.

REFERENCES

Zill, D. G., & Wright, W. S. (2014). *Advanced Engineering Mathematics* (5th ed.). Burlington, MA: Jones & Bartlett Learning.