Lecture XIII

Variation of Parameters

The method of undetermined coefficients only works for linear nonhomogeneous nth-order ODEs with constant coefficients and for particular types of nonhomogeneous functions $g(x)$. Here we will look at a more general method called variation of parameters which works for any linear nonhomogeneous nth-order ODE. Let's begin by looking at the linear second-order ODE in standard form

$$y'' + P(x)y' + Q(x)y = f(x) \tag{1}$$

with homogeneous solution $y_h = c_1y_1(x) + c_2y_2(x)$. We will attempt to seek a solution of the form

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x). \tag{2}$$

Inserting this expression into the ODE gives

$$\begin{aligned} y_p'' + P(x)y_p' + Q(x)y_p = u_1\underbrace{[y_1'' + Py_1' + Qy_1]}_{=0} + u_2\underbrace{[y_2'' + Py_2' + Qy_2]}_{=0} \\ + y_1u_1'' + u_1'y_1' + y_2u_2'' + u_2'y_2' + P[y_1u_1' + y_2u_2'] + y_1'u_1' + y_2'u_2' \\ = \frac{\mathrm{d}}{\mathrm{d}x}[y_1u_1' + y_2u_2'] + P[y_1u_1' + y_2u_2'] + y_1'u_1' + y_2'u_2' = f(x). \end{aligned}$$

We see therefore that if we could find a u_1 and u_2 such that $y_1u_1' + y_2u_2' = 0$ and so $y_1'u_1' + y_2'u_2' = f(x)$, we can obtain our particular solution. This can be neatly obtained by looking at these two equations as a system

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f(x) \end{bmatrix}, \tag{3}$$

which could be solved using Cramer's rule

$$u_1' = \frac{W_1}{W} \quad \text{and} \quad u_2' = \frac{W_2}{W}, \tag{4}$$

where the symbol W is used to denote the determinant of the matrix since it is essentially the Wronskian of y_1 and y_2, and W_j is used to denote the determinant with the jth column replaced by the right-hand side of the equation. Upon integrating to solve for u_1 and u_2, we will have the particular solution $y_p = u_1 y_1 + u_2 y_2$.

The result could be completely generalized to the nonhomogeneous linear nth-order ODE in standard form

$$y^{(n)} + P_{n-1}(x)y^{(n-1)} + \cdots + P_1(x)y' + P_0(x)y = f(x) \tag{5}$$

with homogeneous solution

$$y_h = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n.$$

We assume a particular solution of the form

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x) + \cdots + u_n(x)y_n(x), \tag{6}$$

which upon substitution and use of similar assumptions gives the linear system

$$\begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \\ \vdots \\ u_n' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ f(x) \end{bmatrix}, \tag{7}$$

whose solution is given by

$$u_1' = \frac{W_1}{W}, \quad u_2' = \frac{W_2}{W}, \quad \cdots, \quad u_n' = \frac{W_n}{W}. \tag{8}$$

From here, we would just need to integrate to solve for each u_j and substitute it into our form for the particular solution y_p.

EXAMPLE

$$\text{Solve} \quad 2y'' - 8y' + 26y = 2e^{-2x}.$$

We must first obtain the homogeneous solution, namely $y_h = e^{2x}\left[c_1 \cos(3x) + c_2 \sin(3x)\right]$.

Now, putting the equation in standard form we have

$$y'' - 4y' + 13y = e^{-2x}.$$

We will assume the form $y_p = u_1 y_1 + u_2 y_2$ for the particular solution. We can now solve for u_1 and u_2 by first computing the determinants

$$W = \begin{vmatrix} e^{2x}\cos(3x) & e^{2x}\sin(3x) \\ 2e^{2x}\cos(3x) - 3e^{2x}\sin(3x) & 2e^{2x}\sin(3x) + 3e^{2x}\cos(3x) \end{vmatrix} = 3e^{4x},$$

$$W_1 = \begin{vmatrix} 0 & e^{2x}\sin(3x) \\ e^{-2x} & 2e^{2x}\sin(3x) + 3e^{2x}\cos(3x) \end{vmatrix} = -\sin(3x),$$

$$W_2 = \begin{vmatrix} e^{2x}\cos(3x) & 0 \\ 2e^{2x}\cos(3x) - 3e^{2x}\sin(3x) & e^{-2x} \end{vmatrix} = \cos(3x).$$

We now have

$$u_1' = \frac{W_1}{W} = -\frac{1}{3}e^{-4x}\sin(3x) \quad \text{and} \quad u_2' = \frac{W_2}{W} = \frac{1}{3}e^{-4x}\cos(3x),$$

and so upon integration

$$u_1 = \frac{1}{75}e^{-4x}\left[4\sin(3x) + 3\cos(3x)\right] \quad \text{and} \quad u_2 = \frac{1}{75}e^{-4x}\left[3\sin(3x) - 4\cos(3x)\right].$$

The general solution of the ODE is therefore

$$y = e^{2x}\left[c_1\cos(3x) + c_2\sin(3x)\right] + \frac{1}{25}e^{-2x}.$$

* Note that we could have used the method of undetermined coefficients for this problem.

Lecture Problems (§3.5): 3, 20

Tutorial Problems (§3.5): 2, 6, 13, 21

Suggested Problems (§3.5): 1, 5, 7, 19

BONUS NOTES

None.

REFERENCES

Zill, D. G., & Wright, W. S. (2014). *Advanced Engineering Mathematics* (5th ed.). Burlington, MA: Jones & Bartlett Learning.