Lecture XII

Undetermined Coefficients

In the case of nth-order linear ODEs, we have thus far only looked at obtaining the homogeneous solution. In this lecture, we will begin looking at how to obtain particular solutions when the differential equation is nonhomogeneous. Given the nonhomogenous linear nth-order ODE with constant coefficients

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = g(x), \tag{1}$$

one way to determine the particular solution y_p is to look at the functional form of $g(x)$ and assume a similar form for y_p. This is summarized in the table below.

$g(x)$	Form of y_p
1. $c_n x^n + c_{n-1} x^{n-1} + \cdots + c_2 x^2 + c_1 x + c_0$	$A_n x^n + A_{n-1} x^{n-1} + \cdots + A_2 x^2 + A_1 x + A_0$
2. $c\cos(\alpha x)$	$A\cos(\alpha x) + B\sin(\alpha x)$
3. $c\sin(\alpha x)$	$A\cos(\alpha x) + B\sin(\alpha x)$
4. $ce^{\alpha x}$	$Ae^{\alpha x}$
5. $ce^{\alpha x}\cos(\beta x)$	$Ae^{\alpha x}\cos(\beta x) + Be^{\alpha x}\sin(\beta x)$
6. $ce^{\alpha x}\sin(\beta x)$	$Ae^{\alpha x}\cos(\beta x) + Be^{\alpha x}\sin(\beta x)$
7. $\left(\sum_{j=0}^{n} c_j x^j\right)\cos(\alpha x)$	$\left(\sum_{j=0}^{n} A_j x^j\right)\cos(\alpha x) + \left(\sum_{j=0}^{n} B_j x^j\right)\sin(\alpha x)$
8. $\left(\sum_{j=0}^{n} c_j x^j\right)\sin(\alpha x)$	$\left(\sum_{j=0}^{n} A_j x^j\right)\cos(\alpha x) + \left(\sum_{j=0}^{n} B_j x^j\right)\sin(\alpha x)$
9. $\left(\sum_{j=0}^{n} c_j x^j\right)e^{\alpha x}$	$\left(\sum_{j=0}^{n} A_j x^j\right)e^{\alpha x}$
10. $\left(\sum_{j=0}^{n} c_j x^j\right)e^{\alpha x}\cos(\beta x)$	$\left(\sum_{j=0}^{n} A_j x^j\right)e^{\alpha x}\cos(\beta x) + \left(\sum_{j=0}^{n} B_j x^j\right)e^{\alpha x}\sin(\beta x)$
11. $\left(\sum_{j=0}^{n} c_j x^j\right)e^{\alpha x}\sin(\beta x)$	$\left(\sum_{j=0}^{n} A_j x^j\right)e^{\alpha x}\cos(\beta x) + \left(\sum_{j=0}^{n} B_j x^j\right)e^{\alpha x}\sin(\beta x)$

We have to be careful however, if the assumed form of a particular solution matches a solution in the fundamental set of the homogeneous solution, then we must multiply the particular solution by x.

EXAMPLE

$$\text{Solve} \quad y'' - 10y' + 25y = 50x^2 + 13 - 2e^{5x}.$$

We must first find the homogeneous solution (i.e. the solution to $y'' - 10y' + 25y = 0$). By inspection, the auxiliary equation is $m^2 - 10m + 25 = 0$ giving a double root for m, namely $m_{1,2} = 5$. The homogeneous solution is therefore

$$y_h = c_1 e^{5x} + c_2 x e^{5x}.$$

For the particular solution, we will look at each part of $g(x)$ and develop the assumed form, i.e.

$$y'' - 10y' + 25y = \underbrace{50x^2 + 13}_{y_{p_1} = Ax^2 + Bx + C} - \underbrace{2e^{5x}}_{y_{p_2} = De^{5x}}.$$

Looking at the two particular solutions, we see that $y_{p_1} = Ax^2 + Bx + C$ is not repeated in y_h and so it can be kept this way. On the other hand, the particular solution $y_{p_2} = De^{5x}$ is repeated in y_h, so we must multiply by x, however $y_{p_2} = Dxe^{5x}$ is also seen in y_h, so we multiply by x again giving $y_{p_2} = Dx^2e^{5x}$. The assumed form of the particular solution is therefore

$$y_p = Ax^2 + Bx + C + Dx^2e^{5x}.$$

All that is left now is to determine the constants in the particular solution. This is done by plugging it into the original equation.

$$\begin{gathered} y_p'' - 10y_p' + 25y_p = 50x^2 + 13 - 2e^{5x} \\ 25Ax^2 + (25B - 20A)x + (2A - 10B + 25C) + 2De^{5x} = 50x^2 + 13 - 2e^{5x}. \end{gathered}$$

Comparing each side of the equation, we obtain four equations in four unknowns

$$\begin{aligned} x^2 &\quad \rightarrow \quad 25A = 50 \\ x &\quad \rightarrow \quad 25B - 20A = 0 \\ c &\quad \rightarrow \quad 2A - 10B + 25C = 13 \\ e^{5x} &\quad \rightarrow \quad 2D = -2, \end{aligned}$$

the solution is then $A = 2$, $B = 8/5$, $C = 1$, and $D = -1$. The general solution of the ODE is therefore

$$y = c_1 e^{5x} + c_2 x e^{5x} + 2x^2 + \frac{8}{5}x + 1 - x^2 e^{5x}.$$

Lecture Problems (§3.4): 3, 13, 30

Tutorial Problems (§3.4): 2, 7, 21, 28, 33

Suggested Problems (§3.4): 1, 5, 11, 15, 23, 29, 31

BONUS NOTES

None.

REFERENCES

Zill, D. G., & Wright, W. S. (2014). *Advanced Engineering Mathematics* (5th ed.). Burlington, MA: Jones & Bartlett Learning.