Lecture IV

First-Order Linear Equations

For our second type of ODE, we will consider first-order linear equations of the form

$$a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = g(x). \tag{1}$$

For simplicity, we will consider the ODE in standard form, that is with $a_1(x)$ divided out on both sides, giving the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x). \tag{2}$$

Thinking deeply about such a differential equation, if we multiplied the equation by some function $i(x)$, we may be able to use the product rule (in reverse) to our advantage (a common calculus trick), namely to reduce the equation in the following way

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y &= Q(x) \\ i(x)\left(\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y\right) &= i(x)Q(x) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left[I(x)y\right] &= i(x)Q(x) \end{aligned}$$

Expanding the last equation gives

$$I(x)\frac{\mathrm{d}y}{\mathrm{d}x} + y\frac{\mathrm{d}I(x)}{\mathrm{d}x} = i(x)Q(x),$$

which if we now compare this equation to the original equation, i.e.

$$\begin{aligned} i(x)\left(\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y\right) &= i(x)Q(x) \\ I(x)\frac{\mathrm{d}y}{\mathrm{d}x} + y\frac{\mathrm{d}I(x)}{\mathrm{d}x} &= i(x)Q(x) \end{aligned}$$

we see that the following must hold

$$I(x) = i(x)$$
$$\frac{\mathrm{d}I(x)}{\mathrm{d}x} = I(x)P(x).$$

The equation above is in fact a separable equation with solution

$$I(x) = ce^{\int P(x)\mathrm{d}x}.$$

Coming back to the original equation, we can use $I(x) = e^{\int p(x)\mathrm{d}x}$ as the multiplicative factor, called the **integrating factor**. The constant in $I(x)$ could be ignored since it will be present throughout the equation and can therefore be divided out. We proceed as follows

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)$$
$$I(x)\frac{\mathrm{d}y}{\mathrm{d}x} + I(x)P(x)y = I(x)Q(x)$$
$$e^{\int P(x)\mathrm{d}x}\frac{\mathrm{d}y}{\mathrm{d}x} + e^{\int P(x)\mathrm{d}x}P(x)y = e^{\int P(x)\mathrm{d}x}Q(x).$$

Owing to the product rule then, we can simplify the left-hand side to give

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[ye^{\int P(x)\mathrm{d}x}\right] = e^{\int P(x)\mathrm{d}x}Q(x). \tag{3}$$

The solution is then given by the integral of both sides with respect to x. For completeness, the result is given below with the constant of integration c explicitly shown, however you should not memorize the equation below but rather be able to understand and apply the method of solution.

$$y = \frac{\int e^{\int P(x)\mathrm{d}x}Q(x)\mathrm{d}x}{e^{\int P(x)\mathrm{d}x}} + ce^{-\int P(x)\mathrm{d}x} \tag{4}$$

EXAMPLE

$$\text{Solve} \quad \csc(x)\frac{\mathrm{d}y}{\mathrm{d}x} - y = -1.$$

We must first put the equation in standard form by dividing through by $\csc(x)$

$$\frac{\mathrm{d}y}{\mathrm{d}x} - \sin(x)y = -\sin(x).$$

In this form, $P(x) = -\sin(x)$ and the integrating factor is

$$I(x) = e^{-\int \sin(x)\mathrm{d}x} = e^{\cos(x)}.$$

We can now multiply the equation through by the integrating factor and group the left-hand side into the derivative giving

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[ye^{\cos(x)}\right] = -\sin(x)e^{cos(x)}.$$

Integrating both sides with respect to x and solving for y gives the solution

$$y = 1 + ce^{-\cos(x)}.$$

Sometimes, a first-order linear equation may not be linear in one variable, but is linear in the other. For example,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3}{y^2x + \sin(y)e^y}$$

is clearly not separable and is nonlinear in y. However, if were to express the equation with x as the dependent variable,

$$3\frac{\mathrm{d}x}{\mathrm{d}y} - y^2x = \sin(y)e^y,$$

we get a linear equation that could be solved with the method described in this lecture.

Lecture Problems (§2.3): 15, 22

Tutorial Problems (§2.3): 16, 20, 24, 29

Suggested Problems (§2.3): 7, 9, 19, 23

BONUS NOTES

The method of solving linear first-order equations ultimately boils down to a procedure. Can the equation be expressed in the form below?

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)$$

If not, does the equation become linear when we interchange the dependent and independent variables?

1) Compute the integrating factor.

$$I(x) = e^{\int P(x)\mathrm{d}x}$$

2) Substitute $I(x)$ into the simplified form of the equation.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[I(x)y\right] = I(x)Q(x)$$

3) Integrate both sides (don't forget the constant of integration on the right-hand side).

4) Solve for y and simplify the result.

REFERENCES

Zill, D. G., & Wright, W. S. (2014). *Advanced Engineering Mathematics* (5th ed.). Burlington, MA: Jones & Bartlett Learning.